Chapter 18

Simultaneous-Equation Models

In this and the following two chapters we discuss the simultaneous-equation models. In particular, we discuss their special features, their estimation, and some of the statistical problems associated with them.

18.1 The Nature of Simultaneous-Equation Models

In **Parts 1** to **3** of this text we were concerned exclusively with single-equation models, i.e., models in which there was a single dependent variable Y and one or more explanatory variables, the X's. In such models the emphasis was on estimating and/or predicting the average value of Y conditional upon the fixed values of the X variables. The cause-and-effect relationship, if any, in such models therefore ran from the X's to the Y.

But in many situations, such a one-way or unidirectional cause-and-effect relationship is not meaningful. This occurs if Y is determined by the X's, and some of the X's are, in turn, determined by Y. In short, there is a two-way, or simultaneous, relationship between Y and (some of) the X's, which makes the distinction between *dependent* and *explanatory* variables of dubious value. It is better to lump together a set of variables that can be determined simultaneously by the remaining set of variables—precisely what is done in simultaneous-equation models. In such models there is more than one equation—one for each of the *mutually,* or *jointly,* dependent or **endogenous variables.**[1] And unlike the single-equation models, in the simultaneous-equation models one may not estimate the parameters of a single equation without taking into account information provided by other equations in the system.

What happens if the parameters of each equation are estimated by applying, say, the method of ordinary least squares (OLS), disregarding other equations in the system? Recall that one of the crucial assumptions of the method of OLS is that the explanatory X variables are either nonstochastic or, if stochastic (random), distributed independently of the stochastic disturbance term. If neither of these conditions is met, then, as shown later, the least-squares estimators are not only biased but also inconsistent; that is, as the sample size

[1]In the context of the simultaneous-equation models, the jointly dependent variables are called **endogenous variables** and the variables that are truly nonstochastic or can be so regarded are called the **exogenous,** or **predetermined, variables.** (More on this in Chapter 19.)

increases indefinitely, the estimators do not converge to their true (population) values. Thus, in the following hypothetical system of equations,[2]

$$Y_{1i} = \beta_{10} + \beta_{12}Y_{2i} + \gamma_{11}X_{1i} + u_{1i} \quad \textbf{(18.1.1)}$$

$$Y_{2i} = \beta_{20} + \beta_{21}Y_{1i} + \gamma_{21}X_{1i} + u_{2i} \quad \textbf{(18.1.2)}$$

where Y_1 and Y_2 are mutually dependent, or endogenous, variables and X_1 is an exogenous variable and where u_1 and u_2 are the stochastic disturbance terms, the variables Y_1 and Y_2 are both stochastic. Therefore, unless it can be shown that the stochastic explanatory variable Y_2 in Eq. (18.1.1) is distributed independently of u_1 and the stochastic explanatory variable Y_1 in Eq. (18.1.2) is distributed independently of u_2, application of the classical OLS to these equations individually will lead to inconsistent estimates.

In the remainder of this chapter we give a few examples of simultaneous-equation models and show the bias involved in the direct application of the least-squares method to such models. After discussing the so-called identification problem in Chapter 19, in Chapter 20 we discuss some of the special methods developed to handle the simultaneous-equation models.

18.2 Examples of Simultaneous-Equation Models

EXAMPLE 18.1
Demand-and-Supply Model

As is well known, the price P of a commodity and the quantity Q sold are determined by the intersection of the demand-and-supply curves for that commodity. Thus, assuming for simplicity that the demand-and-supply curves are linear and adding the stochastic disturbance terms u_1 and u_2, we may write the empirical demand-and-supply functions as:

Demand function: $$Q_t^d = \alpha_0 + \alpha_1 P_t + u_{1t} \qquad \alpha_1 < 0 \quad \textbf{(18.2.1)}$$

Supply function: $$Q_t^s = \beta_0 + \beta_1 P_t + u_{2t} \qquad \beta_1 > 0 \quad \textbf{(18.2.2)}$$

Equilibrium condition: $$Q_t^d = Q_t^s$$

where Q^d = quantity demanded
Q^s = quantity supplied
t = time

and the α's and β's are the parameters. A priori, α_1 is expected to be negative (downward-sloping demand curve), and β_1 is expected to be positive (upward-sloping supply curve).

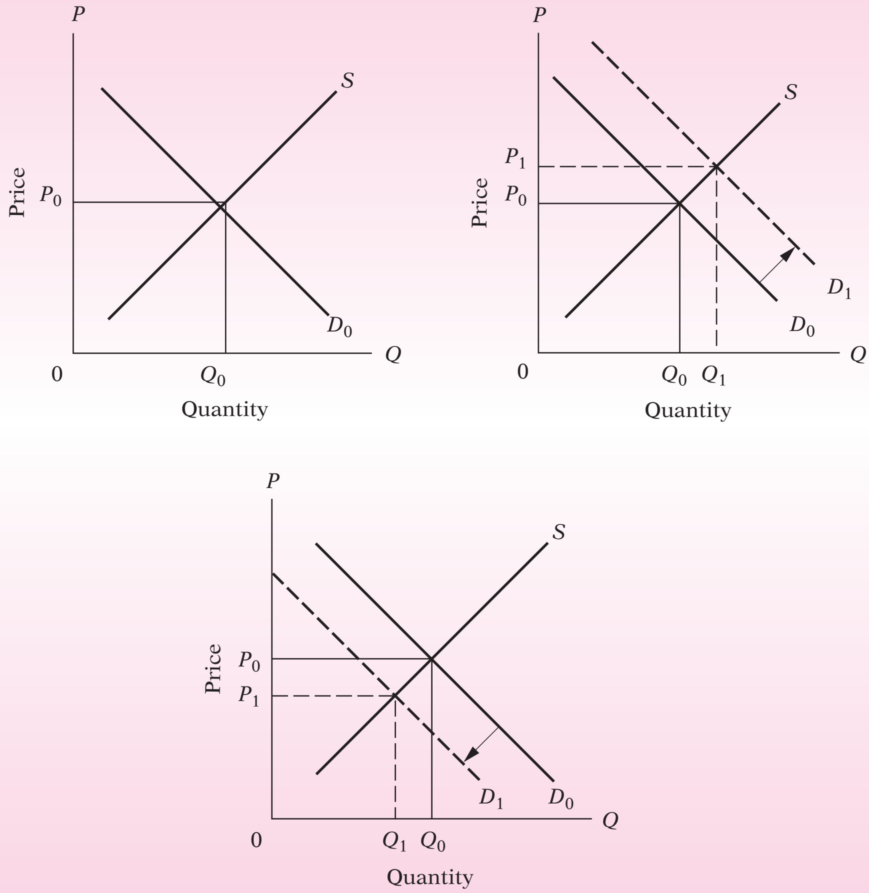
Now it is not too difficult to see that P and Q are jointly dependent variables. If, for example, u_{1t} in Eq. (18.2.1) changes because of changes in other variables affecting Q_t^d (such as income, wealth, and tastes), the demand curve will shift upward if u_{1t} is positive and downward if u_{1t} is negative. These shifts are shown in Figure 18.1.

As the figure shows, a shift in the demand curve changes both P and Q. Similarly, a change in u_{2t} (because of strikes, weather, import or export restrictions, etc.) will shift the supply curve, again affecting both P and Q. Because of this simultaneous dependence between Q and P, u_{1t} and P_t in Eq. (18.2.1) and u_{2t} and P_t in Eq. (18.2.2) cannot be independent. Therefore, a regression of Q on P as in Eq. (18.2.1) would violate an important assumption of the classical linear regression model, namely, the assumption of no correlation between the explanatory variable(s) and the disturbance term.

[2]These economical but self-explanatory notations will be generalized to more than two equations in Chapter 19.

EXAMPLE 18.1
(Continued)

FIGURE 18.1 Interdependence of price and quantity.



EXAMPLE 18.2
Keynesian Model of Income Determination

Consider the simple Keynesian model of income determination:

$$\textit{Consumption function:} \quad C_t = \beta_0 + \beta_1 Y_t + u_t \quad 0 < \beta_1 < 1 \qquad \textbf{(18.2.3)}$$

$$\textit{Income identity:} \quad Y_t = C_t + I_t (= S_t) \qquad \textbf{(18.2.4)}$$

where
C = consumption expenditure
Y = income
I = investment (assumed exogenous)
S = savings
t = time
u = stochastic disturbance term
β_0 and β_1 = parameters

The parameter β_1 is known as the *marginal propensity to consume* (MPC) (the amount of extra consumption expenditure resulting from an extra dollar of income). From economic theory, β_1 is expected to lie between 0 and 1. Equation (18.2.3) is the (stochastic) consumption function; and Eq. (18.2.4) is the national income identity, signifying that total income is equal to total consumption expenditure plus total investment expenditure, it

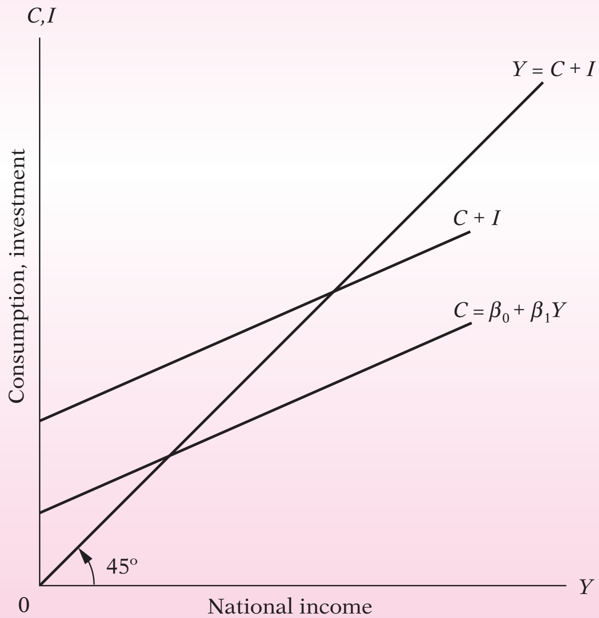
(Continued)

EXAMPLE 18.2 *(Continued)*

being understood that total investment expenditure is equal to total savings. Diagrammatically, we have Figure 18.2.

From the postulated consumption function and Figure 18.2 it is clear that C and Y are interdependent and that Y_t in Eq. (18.2.3) is not expected to be independent of the disturbance term because when u_t shifts (because of a variety of factors subsumed in the error term), then the consumption function also shifts, which, in turn, affects Y_t. Therefore, once again the classical least-squares method is inapplicable to Eq. (18.2.3). If applied, the estimators thus obtained will be inconsistent, as we shall show later.

FIGURE 18.2
Keynesian model of income determination.



EXAMPLE 18.3 *Wage–Price Models*

Consider the following Phillips-type model of money-wage and price determination:

$$\dot{W}_t = \alpha_0 + \alpha_1 \text{UN}_t + \alpha_2 \dot{P}_t + u_{1t} \qquad \textbf{(18.2.5)}$$

$$\dot{P}_t = \beta_0 + \beta_1 \dot{W}_t + \beta_2 \dot{R}_t + \beta_3 \dot{M}_t + u_{2t} \qquad \textbf{(18.2.6)}$$

where $\dot{W}$ = rate of change of money wages
UN = unemployment rate, %
$\dot{P}$ = rate of change of prices
$\dot{R}$ = rate of change of cost of capital
$\dot{M}$ = rate of change of price of imported raw material
t = time
u_1, u_2 = stochastic disturbances

Since the price variable $\dot{P}$ enters into the wage equation and the wage variable $\dot{W}$ enters into the price equation, the two variables are jointly dependent. Therefore, these stochastic explanatory variables are expected to be correlated with the relevant stochastic disturbances, once again rendering the classical OLS method inapplicable to estimate the parameters of the two equations individually.

EXAMPLE 18.4
The IS Model of Macroeconomics

The celebrated IS, or goods market equilibrium, model of macroeconomics[3] in its non-stochastic form can be expressed as:

Consumption function: $C_t = \beta_0 + \beta_1 Y_{dt} \quad 0 < \beta_1 < 1$ **(18.2.7)**

Tax function: $T_t = \alpha_0 + \alpha_1 Y_t \quad 0 < \alpha_1 < 1$ **(18.2.8)**

Investment function: $I_t = \gamma_0 + \gamma_1 r_t$ **(18.2.9)**

Definition: $Y_{dt} = Y_t - T_t$ **(18.2.10)**

Government expenditure: $G_t = \bar{G}$ **(18.2.11)**

National income identity: $Y_t = C_t + I_t + G_t$ **(18.2.12)**

where Y = national income
C = consumption spending
I = planned or desired net investment
$\bar{G}$ = given level of government expenditure
T = taxes
Y_d = disposable income
r = interest rate

If you substitute Eqs. (18.2.10) and (18.2.8) into Eq. (18.2.7) and substitute the resulting equation for C and Eqs. (18.2.9) and (18.2.11) into Eq. (18.2.12), you should obtain the IS equation:

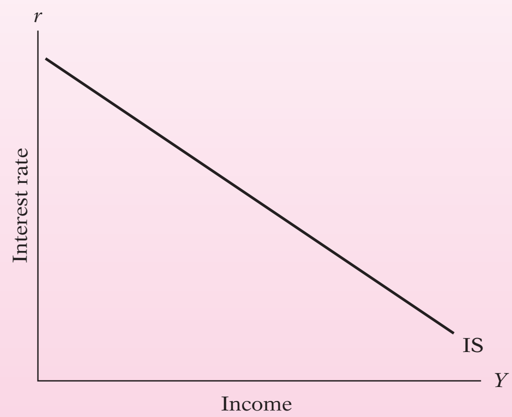
$$Y_t = \pi_0 + \pi_1 r_t \tag{18.2.13}$$

where

$$\pi_0 = \frac{\beta_0 - \alpha_0\beta_1 + \gamma_0 + \bar{G}}{1 - \beta_1(1 - \alpha_1)}$$
$$\pi_1 = \frac{1}{1 - \beta_1(1 - \alpha_1)} \tag{18.2.14}$$

Equation (18.2.13) is the equation of the IS, or goods market equilibrium, that is, it gives the combinations of the interest rate and level of income such that the goods market clears or is in equilibrium. Geometrically, the IS curve is shown in Figure 18.3.

FIGURE 18.3
The IS curve.



(*Continued*)

[3]"The goods market equilibrium schedule, or IS schedule, shows combinations of interest rates and levels of output such that planned spending equals income." See Rudiger Dornbusch and Stanley Fischer, *Macroeconomics,* 3d ed., McGraw-Hill, New York, 1984, p. 102. Note that for simplicity we have assumed away the foreign trade sector.

EXAMPLE 18.4
(Continued)

What would happen if we were to estimate, say, the consumption function (18.2.7) in isolation? Could we obtain unbiased and/or consistent estimates of β_0 and β_1? Such a result is unlikely because consumption depends on disposable income, which depends on national income Y, but the latter depends on r and $\bar{G}$ as well as the other parameters entering in π_0. Therefore, unless we take into account all these influences, a simple regression of C on Y_d is bound to give biased and/or inconsistent estimates of β_0 and β_1.

EXAMPLE 18.5
The LM Model

The other half of the famous IS–LM paradigm is the LM, or money market equilibrium, relation, which gives the combinations of the interest rate and level of income such that the money market is cleared, that is, the demand for money is equal to its supply. Algebraically, the model, in the nonstochastic form, may be expressed as:

$$\text{Money demand function:} \quad M_t^d = a + bY_t - cr_t \tag{18.2.15}$$

$$\text{Money supply function:} \quad M_t^s = \bar{M} \tag{18.2.16}$$

$$\text{Equilibrium condition:} \quad M_t^d = M_t^s \tag{18.2.17}$$

where Y = income, r = interest rate, and $\bar{M}$ = assumed level of money supply, say, determined by the Fed.

Equating the money demand and supply functions and simplifying, we obtain the LM equation:

$$Y_t = \lambda_0 + \lambda_1 \bar{M} + \lambda_2 r_t \tag{18.2.18}$$

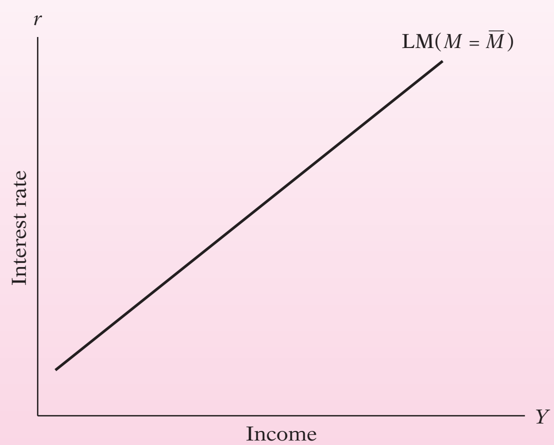
where

$$\begin{aligned} \lambda_0 &= -a/b \\ \lambda_1 &= 1/b \\ \lambda_2 &= c/b \end{aligned} \tag{18.2.19}$$

For a given $M = \bar{M}$, the LM curve representing the relation (18.2.18) is as shown in Figure 18.4.

The IS and LM curves show, respectively, that a whole array of interest rates is consistent with goods market equilibrium and a whole array of interest rates is compatible with equilibrium in the money market. Of course, only one interest rate and one level of income will be consistent simultaneously with the two equilibria. To obtain these, all that needs to be done is to equate Eqs. (18.2.13) and (18.2.18). In Exercise 18.4 you are asked to show the level of the interest rate and income that is simultaneously compatible with the goods and money market equilibrium.

FIGURE 18.4
The LM curve.



EXAMPLE 18.6
Econometric Models

An extensive use of simultaneous-equation models has been made in the econometric models built by several econometricians. An early pioneer in this field was Professor Lawrence Klein of the Wharton School of the University of Pennsylvania. His initial model, known as **Klein's model I,** is as follows:

$$\begin{aligned}
&\textit{Consumption function:} && C_t = \beta_0 + \beta_1 P_t + \beta_2 (W + W')_t + \beta_3 P_{t-1} + u_{1t} \\
&\textit{Investment function:} && I_t = \beta_4 + \beta_5 P_t + \beta_6 P_{t-1} + \beta_7 K_{t-1} + u_{2t} \\
&\textit{Demand for labor:} && W_t = \beta_8 + \beta_9 (Y + T - W')_t \\
& && \quad + \beta_{10} (Y + T - W')_{t-1} + \beta_{11} t + u_{3t} \\
&\textit{Identity:} && Y_t + T_t = C_t + I_t + G_t \\
&\textit{Identity:} && Y_t = W'_t + W_t + P_t \\
&\textit{Identity:} && K_t = K_{t-1} + I_t
\end{aligned} \tag{18.2.20}$$

where
C = consumption expenditure
I = investment expenditure
G = government expenditure
P = profits
W = private wage bill
W' = government wage bill
K = capital stock
T = taxes
Y = income after tax
t = time
u_1, u_2, and u_3 = stochastic disturbances[4]

In the preceding model the variables *C, I, W, Y, P,* and *K* are treated as jointly dependent, or endogenous, variables and the variables P_{t-1}, K_{t-1}, and Y_{t-1} are treated as predetermined.[5] In all, there are six equations (including the three identities) to study the interdependence of six endogenous variables.

In Chapter 20 we shall see how such econometric models are estimated. For the time being, note that because of the interdependence among the endogenous variables, in general they are not independent of the stochastic disturbance terms, which therefore makes it inappropriate to apply the method of OLS to an individual equation in the system. As shown in Section 18.3, the estimators thus obtained are inconsistent; they do not converge to their true population values even when the sample size is very large.

18.3 The Simultaneous-Equation Bias: Inconsistency of OLS Estimators

As stated previously, the method of least squares may not be applied to estimate a single equation embedded in a system of simultaneous equations if one or more of the explanatory variables are correlated with the disturbance term in that equation because the estimators thus obtained are inconsistent. To show this, let us revert to the simple Keynesian

[4] L. R. Klein, *Economic Fluctuations in the United States, 1921–1941,* John Wiley & Sons, New York, 1950.

[5] The model builder will have to specify which of the variables in a model are endogenous and which are predetermined. K_{t-1} and Y_{t-1} are predetermined because at time t their values are known. (More on this in Chapter 19.)

model of income determination given in Example 18.2. Suppose that we want to estimate the parameters of the consumption function (18.2.3). Assuming that $E(u_t) = 0$, $E(u_t^2) = \sigma^2$, $E(u_t u_{t+j}) = 0$ (for $j \neq 0$), and $\text{cov}\,(I_t, u_t) = 0$, which are the assumptions of the classical linear regression model, we first show that Y_t and u_t in (18.2.3) are correlated and then prove that $\hat{\beta}_1$ is an inconsistent estimator of β_1.

To prove that Y_t and u_t are correlated, we proceed as follows. Substitute Eq. (18.2.3) into Eq. (18.2.4) to obtain

$$Y_t = \beta_0 + \beta_1 Y_t + u_t + I_t$$

that is,

$$Y_t = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t + \frac{1}{1-\beta_1} u_t \tag{18.3.1}$$

Now

$$E(Y_t) = \frac{\beta_0}{1-\beta_1} + \frac{1}{1-\beta_1} I_t \tag{18.3.2}$$

where use is made of the fact that $E(u_t) = 0$ and that I_t being exogenous, or predetermined (because it is fixed in advance), has as its expected value I_t.

Therefore, subtracting Eq. (18.3.2) from Eq. (18.3.1) results in

$$Y_t - E(Y_t) = \frac{u_t}{1-\beta_1} \tag{18.3.3}$$

Moreover,

$$u_t - E(u_t) = u_t \qquad \text{(Why?)} \tag{18.3.4}$$

whence

$$\begin{aligned} \text{cov}\,(Y_t, u_t) &= E[Y_t - E(Y_t)][u_t - E(u_t)] \\ &= \frac{E\left(u_t^2\right)}{1-\beta_1} \qquad \text{from Eqs. (18.3.3) and (18.3.4)} \\ &= \frac{\sigma^2}{1-\beta_1} \end{aligned} \tag{18.3.5}$$

Since σ^2 is positive by assumption (why?), the covariance between Y and u given in Eq. (18.3.5) is bound to be different from zero.[6] As a result, Y_t and u_t in Eq. (18.2.3) are expected to be correlated, which violates the assumption of the classical linear regression model that the disturbances are independent or at least uncorrelated with the explanatory variables. As noted previously, the OLS estimators in this situation are inconsistent.

To show that the OLS estimator $\hat{\beta}_1$ is an inconsistent estimator of β_1 because of correlation between Y_t and u_t, we proceed as follows:

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum (C_t - \bar{C})(Y_t - \bar{Y})}{\sum (Y_t - \bar{Y})^2} \\ &= \frac{\sum c_t y_t}{\sum y_t^2} \\ &= \frac{\sum C_t y_t}{\sum y_t^2} \end{aligned} \tag{18.3.6}$$

[6]It will be greater than zero as long as β_1, the MPC, lies between 0 and 1, and it will be negative if β_1 is greater than unity. Of course, a value of MPC greater than unity would not make much economic sense. In reality therefore the covariance between Y_t and u_t is expected to be positive.

where the lowercase letters, as usual, indicate deviations from the (sample) mean values. Substituting for C_t from Eq. (18.2.3), we obtain

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum(\beta_0 + \beta_1 Y_t + u_t)y_t}{\sum y_t^2} \\ &= \beta_1 + \frac{\sum y_t u_t}{\sum y_t^2}\end{aligned} \tag{18.3.7}$$

where in the last step use is made of the fact that $\sum y_t = 0$ and $(\sum Y_t y_t / \sum y_t^2) = 1$ (why?).

If we take the expectation of Eq. (18.3.7) on both sides, we obtain

$$E(\hat{\beta}_1) = \beta_1 + E\left[\frac{\sum y_t u_t}{\sum y_t^2}\right] \tag{18.3.8}$$

Unfortunately, we cannot evaluate $E(\sum y_t u_t / \sum y_t^2)$ since the expectations operator is a linear operator. [*Note:* $E(A/B) \neq E(A)/E(B)$.] But intuitively it should be clear that unless the term $(\sum y_t u_t / \sum y_t^2)$ is zero, $\hat{\beta}_1$ is a biased estimator of β_1. But have we not shown in Eq. (18.3.5) that the covariance between Y and u is nonzero and therefore would $\hat{\beta}_1$ not be biased? The answer is, not quite, since cov (Y_t, u_t), a population concept, is not quite $\sum y_t u_t$, which is a sample measure, although as the sample size increases indefinitely the latter will tend toward the former. But if the sample size increases indefinitely, then we can resort to the concept of consistent estimator and find out what happens to $\hat{\beta}_1$ as n, the sample size, increases indefinitely. In short, when we cannot explicitly evaluate the expected value of an estimator, as in Eq. (18.3.8), we can turn our attention to its behavior in the large sample.

Now an estimator is said to be consistent if its **probability limit,**[7] or **plim** for short, is equal to its true (population) value. Therefore, to show that $\hat{\beta}_1$ of Eq. (18.3.7) is inconsistent, we must show that its plim is not equal to the true β_1. Applying the rules of probability limit to Eq. (18.3.7), we obtain:[8]

$$\begin{aligned}\text{plim}\,(\hat{\beta}_1) &= \text{plim}\,(\beta_1) + \text{plim}\left(\frac{\sum y_t u_t}{\sum y_t^2}\right) \\ &= \text{plim}\,(\beta_1) + \text{plim}\left(\frac{\sum y_t u_t / n}{\sum y_t^2 / n}\right) \\ &= \beta_1 + \frac{\text{plim}\,(\sum y_t u_t / n)}{\text{plim}\,(\sum y_t^2 / n)}\end{aligned} \tag{18.3.9}$$

where in the second step we have divided $\sum y_t u_t$ and $\sum y_t^2$ by the total number of observations in the sample n so that the quantities in the parentheses are now the sample covariance between Y and u and the sample variance of Y, respectively.

In words, Eq. (18.3.9) states that the probability limit of $\hat{\beta}_1$ is equal to true β_1 plus the ratio of the plim of the sample covariance between Y and u to the plim of the sample variance of Y. Now as the sample size n increases indefinitely, one would expect the sample covariance between Y and u to approximate the true population covariance $E[Y_t - E(Y_t)][u_t - E(u_t)]$, which from Eq. (18.3.5) is equal to $[\sigma^2/(1 - \beta_1)]$. Similarly, as n tends to infinity, the sample

[7]See **Appendix A** for the definition of probability limit.

[8]As stated in **Appendix A,** the plim of a constant (for example, β_1) is the same constant and the plim of $(A/B) = \text{plim}\,(A)/\text{plim}\,(B)$. Note, however, that $E(A/B) \neq E(A)/E(B)$.

variance of Y will approximate its population variance, say σ_Y^2. Therefore, Eq. (18.3.9) may be written as

$$\begin{aligned}\text{plim}\,(\hat{\beta}_1) &= \beta_1 + \frac{\sigma^2/(1-\beta_1)}{\sigma_Y^2} \\ &= \beta_1 + \frac{1}{1-\beta_1}\left(\frac{\sigma^2}{\sigma_Y^2}\right)\end{aligned} \tag{18.3.10}$$

Given that $0 < \beta_1 < 1$ and that σ^2 and σ_Y^2 are both positive, it is obvious from Eq. (18.3.10) that plim $(\hat{\beta}_1)$ will always be greater than β_1; that is, $\hat{\beta}_1$ will overestimate the true β_1.[9] In other words, $\hat{\beta}_1$ is a biased estimator, and the bias will not disappear no matter how large the sample size.

18.4 The Simultaneous-Equation Bias: A Numerical Example

To demonstrate some of the points made in the preceding section, let us return to the simple Keynesian model of income determination given in Example 18.2 and carry out the following **Monte Carlo** study.[10] Assume that the values of investment I are as shown in column 3 of Table 18.1. Further assume that

$$\begin{aligned} E(u_t) &= 0 \\ E(u_t u_{t+j}) &= 0 \qquad (j \neq 0) \\ \text{var}\,(u_t) &= \sigma^2 = 0.04 \\ \text{cov}\,(u_t, I_t) &= 0 \end{aligned}$$

The u_t thus generated are shown in column 4.

For the consumption function (18.2.3) assume that the values of the true parameters are known and are $\beta_0 = 2$ and $\beta_1 = 0.8$.

From the assumed values of β_0 and β_1 and the generated values of u_t we can generate the values of income Y_t from Eq. (18.3.1), which are shown in column 1 of Table 18.1. Once Y_t are known, and knowing β_0, β_1, and u_t, one can easily generate the values of consumption C_t from Eq. (18.2.3). The C's thus generated are given in column 2.

Since the true β_0 and β_1 are known, and since our sample errors are exactly the same as the "true" errors (because of the way we designed the Monte Carlo study), if we use the data of Table 18.1 to regress C_t on Y_t we should obtain $\beta_0 = 2$ and $\beta_1 = 0.8$, if OLS were unbiased. But from Eq. (18.3.7) we know that this will not be the case if the regressor Y_t and the disturbance u_t are correlated. Now it is not too difficult to verify from our data that the (sample) covariance between Y_t and u_t is $\sum y_t u_t = 3.8$ and that $\sum y_t^2 = 184$. Then, as Eq. (18.3.7) shows, we should have

$$\begin{aligned}\hat{\beta}_1 &= \beta_1 + \frac{\sum y_t u_t}{\sum y_t^2} \\ &= 0.8 + \frac{3.8}{184} \\ &= 0.82065\end{aligned} \tag{18.4.1}$$

That is, $\hat{\beta}_1$ is upward-biased by 0.02065.

[9]In general, however, the direction of the bias will depend on the structure of the particular model and the true values of the regression coefficients.

[10]This is borrowed from Kenneth J. White, Nancy G. Horsman, and Justin B. Wyatt, *SHAZAM: Computer Handbook for Econometrics for Use with Basic Econometrics,* McGraw-Hill, New York, 1985, pp. 131–134.

TABLE 18.1

Y_t (1)	C_t (2)	I_t (3)	u_t (4)
18.15697	16.15697	2.0	−0.3686055
19.59980	17.59980	2.0	−0.8004084E-01
21.93468	19.73468	2.2	0.1869357
21.55145	19.35145	2.2	0.1102906
21.88427	19.48427	2.4	−0.2314535E-01
22.42648	20.02648	2.4	0.8529544E-01
25.40940	22.80940	2.6	0.4818807
22.69523	20.09523	2.6	−0.6095481E-01
24.36465	21.56465	2.8	0.7292983E-01
24.39334	21.59334	2.8	0.7866819E-01
24.09215	21.09215	3.0	−0.1815703
24.87450	21.87450	3.0	−0.2509900E-01
25.31580	22.11580	3.2	−0.1368398
26.30465	23.10465	3.2	0.6092946E-01
25.78235	22.38235	3.4	−0.2435298
26.08018	22.68018	3.4	−0.1839638
27.24440	23.64440	3.6	−0.1511200
28.00963	24.40963	3.6	0.1926739E-02
30.89301	27.09301	3.8	0.3786015
28.98706	25.18706	3.8	−0.2588852E-02

Source: Kenneth J. White, Nancy G. Horsman, and Justin B. Wyatt, *SHAZAM: Computer Handbook for Econometrics for Use with Damodar Gujarati: Basic Econometrics,* September 1985, p. 132.

Now let us regress C_t on Y_t, using the data given in Table 18.1. The regression results are

$$\begin{aligned} \hat{C}_t &= 1.4940 \quad + \quad 0.82065 Y_t \\ \text{se} &= (0.35413) \quad (0.01434) \\ t &= (4.2188) \quad (57.209) \qquad R^2 = 0.9945 \end{aligned} \tag{18.4.2}$$

As expected, the estimated β_1 is precisely the one predicted by Eq. (18.4.1). In passing, note that the estimated β_0 too is biased.

In general, the amount of the bias in $\hat{\beta}_1$ depends on β_1, σ^2 and var (Y) and, in particular, on the degree of covariance between Y and u.[11] As Kenneth White et al. note, "This is what simultaneous equation bias is all about. In contrast to single equation models, we can no longer assume that variables on the right hand side of the equation are uncorrelated with the error term."[12] Bear in mind that this bias remains even in large samples.

In view of the potentially serious consequences of applying OLS in simultaneous-equation models, is there a test of simultaneity that can tell us whether in a given instance we have the simultaneity problem? One version of the **Hausman specification test** can be used for this purpose, which we discuss in Chapter 19.

[11] See Eq. (18.3.5).

[12] Op. cit., pp. 133–134.

Summary and Conclusions

1. In contrast to single-equation models, in simultaneous-equation models more than one dependent, or **endogenous,** variable is involved, necessitating as many equations as the number of endogenous variables.
2. A unique feature of simultaneous-equation models is that the endogenous variable (i.e., regressand) in one equation may appear as an explanatory variable (i.e., regressor) in another equation of the system.
3. As a consequence, such an **endogenous explanatory variable** becomes stochastic and is usually correlated with the disturbance term of the equation in which it appears as an explanatory variable.
4. In this situation the classical OLS method may not be applied because the estimators thus obtained are not consistent, that is, they do not converge to their true population values no matter how large the sample size.
5. The Monte Carlo example presented in the text shows the nature of the bias involved in applying OLS to estimate the parameters of a regression equation in which the regressor is correlated with the disturbance term, which is typically the case in simultaneous-equation models.
6. Since simultaneous-equation models are used frequently, especially in econometric models, alternative estimating techniques have been developed by various authors. These are discussed in Chapter 20, after the topic of the **identification problem** is considered in Chapter 19, a topic logically prior to estimation.

EXERCISES

Questions

18.1. Develop a simultaneous-equation model for the supply of and demand for dentists in the United States. Specify the endogenous and exogenous variables in the model.

18.2. Develop a simple model of the demand for and supply of money in the United States and compare your model with those developed by K. Brunner and A. H. Meltzer* and R. Tiegen.†

18.3. *a.* For the demand-and-supply model of Example 18.1, obtain the expression for the probability limit of $\hat{\alpha}_1$.

b. Under what conditions will this probability limit be equal to the true α_1?

18.4. For the IS-LM model discussed in the text, find the level of interest rate and income that is simultaneously compatible with the goods and money market equilibrium.

18.5. To study the relationship between inflation and yield on common stock, Bruno Oudet‡ used the following model:

$$R_{bt} = \alpha_1 + \alpha_2 R_{st} + \alpha_3 R_{bt-1} + \alpha_4 L_t + \alpha_5 Y_t + \alpha_6 \text{NIS}_t + \alpha_7 I_t + u_{1t}$$

$$R_{st} = \beta_1 + \beta_2 R_{bt} + \beta_3 R_{bt-1} + \beta_4 L_t + \beta_5 Y_t + \beta_6 \text{NIS}_t + \beta_7 E_t + u_{2t}$$

*"Some Further Evidence on Supply and Demand Functions for Money," *Journal of Finance,* vol. 19, May 1964, pp. 240–283.

†"Demand and Supply Functions for Money in the United States," *Econometrica,* vol. 32, no. 4, October 1964, pp. 476–509.

‡Bruno A. Oudet, "The Variation of the Return on Stocks in Periods of Inflation," *Journal of Financial and Quantitative Analysis,* vol. 8, no. 2, March 1973, pp. 247–258.

where L = real per capita monetary base
Y = real per capita income
I = the expected rate of inflation
NIS = a new issue variable
E = expected end-of-period stock returns, proxied by lagged stock price ratios
R_{bt} = bond yield
R_{st} = common stock returns

a. Offer a theoretical justification for this model and see if your reasoning agrees with that of Oudet.

b. Which are the endogenous variables in the model? Which are the exogenous variables?

c. How would you treat the lagged R_{bt}—endogenous or exogenous?

18.6. In their article, "A Model of the Distribution of Branded Personal Products in Jamaica,"* John U. Farley and Harold J. Levitt developed the following model (the personal products considered were shaving cream, skin cream, sanitary napkins, and toothpaste):

$$Y_{1i} = \alpha_1 + \beta_1 Y_{2i} + \beta_2 Y_{3i} + \beta_3 Y_{4i} + u_{1i}$$
$$Y_{2i} = \alpha_2 + \beta_4 Y_{1i} + \beta_5 Y_{5i} + \gamma_1 X_{1i} + \gamma_2 X_{2i} + u_{2i}$$
$$Y_{3i} = \alpha_3 + \beta_6 Y_{2i} + \gamma_3 X_{3i} + u_{3i}$$
$$Y_{4i} = \alpha_4 + \beta_7 Y_{2i} + \gamma_4 X_{4i} + u_{4i}$$
$$Y_{5i} = \alpha_5 + \beta_8 Y_{2i} + \beta_9 Y_{3i} + \beta_{10} Y_{4i} + u_{5i}$$

where Y_1 = percent of stores stocking the product
Y_2 = sales in units per month
Y_3 = index of direct contact with importer and manufacturer for the product
Y_4 = index of wholesale activity in the area
Y_5 = index of depth of brand stocking for the product (i.e., average number of brands of the product stocked by stores carrying the product)
X_1 = target population for the product
X_2 = income per capita in the parish where the area is
X_3 = distance from the population center of gravity to Kingston
X_4 = distance from population center to nearest wholesale town

a. Can you identify the endogenous and exogenous variables in the preceding model?

b. Can one or more equations in the model be estimated by the method of least squares? Why or why not?

18.7. To study the relationship between advertising expenditure and sales of cigarettes, Frank Bass used the following model:†

$$Y_{1t} = \alpha_1 + \beta_1 Y_{3t} + \beta_2 Y_{4t} + \gamma_1 X_{1t} + \gamma_2 X_{2t} + u_{1t}$$
$$Y_{2t} = \alpha_2 + \beta_3 Y_{3t} + \beta_4 Y_{4t} + \gamma_3 X_{1t} + \gamma_4 X_{2t} + u_{2t}$$
$$Y_{3t} = \alpha_3 + \beta_5 Y_{1t} + \beta_6 Y_{2t} + u_{3t}$$
$$Y_{4t} = \alpha_4 + \beta_7 Y_{1t} + \beta_8 Y_{2t} + u_{4t}$$

**Journal of Marketing Research,* November 1968, pp. 362–368.

†"A Simultaneous Equation Regression Study of Advertising and Sales of Cigarettes," *Journal of Marketing Research,* vol. 6, August 1969, pp. 291–300.

where Y_1 = logarithm of sales of filter cigarettes (number of cigarettes) divided by population over age 20
Y_2 = logarithm of sales of nonfilter cigarettes (number of cigarettes) divided by population over age 20
Y_3 = logarithm of advertising dollars for filter cigarettes divided by population over age 20 divided by advertising price index
Y_4 = logarithm of advertising dollars for nonfilter cigarettes divided by population over age 20 divided by advertising price index
X_1 = logarithm of disposable personal income divided by population over age 20 divided by consumer price index
X_2 = logarithm of price per package of nonfilter cigarettes divided by consumer price index

a. In the preceding model the Y's are endogenous and the X's are exogenous. Why does the author assume X_2 to be exogenous?

b. If X_2 is treated as an endogenous variable, how would you modify the preceding model?

18.8. G. Menges developed the following econometric model for the West German economy:*

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 I_t + u_{1t}$$
$$I_t = \beta_3 + \beta_4 Y_t + \beta_5 Q_t + u_{2t}$$
$$C_t = \beta_6 + \beta_7 Y_t + \beta_8 C_{t-1} + \beta_9 P_t + u_{3t}$$
$$Q_t = \beta_{10} + \beta_{11} Q_{t-1} + \beta_{12} R_t + u_{4t}$$

where Y = national income
I = net capital formation
C = personal consumption
Q = profits
P = cost of living index
R = industrial productivity
t = time
u = stochastic disturbances

a. Which of the variables would you regard as endogenous and which as exogenous?

b. Is there any equation in the system that can be estimated by the single-equation least-squares method?

c. What is the reason behind including the variable P in the consumption function?

18.9. L. E. Gallaway and P. E. Smith developed a simple model for the United States economy, which is as follows:†

$$Y_t = C_t + I_t + G_t$$
$$C_t = \beta_1 + \beta_2 \mathrm{YD}_{t-1} + \beta_3 M_t + u_{1t}$$
$$I_t = \beta_4 + \beta_5 (Y_{t-1} - Y_{t-2}) + \beta_6 Z_{t-1} + u_{2t}$$
$$G_t = \beta_7 + \beta_8 G_{t-1} + u_{3t}$$

*G. Menges, "Ein Ökonometriches Modell der Bundesrepublik Deutschland (Vier Strukturgleichungen)," I.F.O. Studien, vol. 5, 1959, pp. 1–22.

†"A Quarterly Econometric Model of the United States," *Journal of American Statistical Association,* vol. 56, 1961, pp. 379–383.

where
Y = gross national product
C = personal consumption expenditure
I = gross private domestic investment
G = government expenditure plus net foreign investment
YD = disposable, or after-tax, income
M = money supply at the beginning of the quarter
Z = property income before taxes
t = time
u_1, u_2, and u_3 = stochastic disturbances

All variables are measured in the first-difference form.

From the quarterly data from 1948–1957, the authors applied the least-squares method to each equation individually and obtained the following results:

$$\hat{C}_t = 0.09 + 0.43\text{YD}_{t-1} + 0.23M_t \qquad R^2 = 0.23$$

$$\hat{I}_t = 0.08 + 0.43(Y_{t-1} - Y_{t-2}) + 0.48Z_t \qquad R^2 = 0.40$$

$$\hat{G}_t = 0.13 + 0.67G_{t-1} \qquad R^2 = 0.42$$

a. How would you justify the use of the single-equation least-squares method in this case?

b. Why are the R^2 values rather low?

Empirical Exercises

18.10. Table 18.2 gives you data on Y (gross domestic product), I (gross private domestic investment), and C (personal consumption expenditure) for the United States for the period 1970–2006. All data are in 1996 billions of dollars. Assume that C is linearly related to Y as in the simple Keynesian model of income determination of Example 18.2. Obtain OLS estimates of the parameters of the consumption function. Save the results for another look at the same data using the methods developed in Chapter 20.

18.11. Using the data given in Exercise 18.10, regress gross domestic investment I on GDP and save the results for further examination in a later chapter.

18.12. Consider the macroeconomics identity

$$C + I = Y \qquad (= \text{GDP})$$

As before, assume that

$$C_t = \beta_0 + \beta_1 Y_t + u_t$$

and, following the **accelerator model** of macroeconomics, let

$$I_t = \alpha_0 + \alpha_1(Y_t - Y_{t-1}) + v_t$$

where u and v are error terms. From the data given in Exercise 18.10, estimate the accelerator model and save the results for further study.

18.13. *Supply and demand for gas.* Table 18.3, found on the textbook website, gives data on some of the variables that determine demand for and supply of gasoline in the U.S. from January 1978 to August 2002.* The variables are: pricegas (cents per

*These data are taken from the website of Stephen J. Schmidt, *Econometrics,* McGraw-Hill, New York, 2005. See www.mhhe.com/economics.

TABLE 18.2 Personal Consumption Expenditure, Gross Private Domestic Investment, and GDP, United States, 1970–2006 (billions of 1996 dollars)

Observation	C	I	Y	Observation	C	I	Y
1970	2,451.9	427.1	3,771.9	1989	4,675.0	926.2	6,981.4
1971	2,545.5	475.7	3,898.6	1990	4,770.3	895.1	7,112.5
1972	2,701.3	532.1	4,105.0	1991	4,778.4	822.2	7,100.5
1973	2,833.8	594.4	4,341.5	1992	4,934.8	889.0	7,336.6
1974	2,812.3	550.6	4,319.6	1993	5,099.8	968.3	7,532.7
1975	2,876.9	453.1	4,311.2	1994	5,290.7	1,099.6	7,835.5
1976	3,035.5	544.7	4,540.9	1995	5,433.5	1,134.0	8,031.7
1977	3,164.1	627.0	4,750.5	1996	5,619.4	1,234.3	8,328.9
1978	3,303.1	702.6	5,015.0	1997	5,831.8	1,387.7	8,703.5
1979	3,383.4	725.0	5,173.4	1998	6,125.8	1,524.1	9,066.9
1980	3,374.1	645.3	5,161.7	1999	6,438.6	1,642.6	9,470.3
1981	3,422.2	704.9	5,291.7	2000	6,739.4	1,735.5	9,817.0
1982	3,470.3	606.0	5,189.3	2001	6,910.4	1,598.4	9,890.7
1983	3,668.6	662.5	5,423.8	2002	7,099.3	1,557.1	10,048.8
1984	3,863.3	857.7	5,813.6	2003	7,295.3	1,613.1	10,301.0
1985	4,064.0	849.7	6,053.7	2004	7,561.4	1,770.2	10,675.8
1986	4,228.9	843.9	6,263.6	2005	7,803.6	1,869.3	11,003.4
1987	4,369.8	870.0	6,475.1	2006	8,044.1	1,919.5	11,319.4
1988	4,546.9	890.5	6,742.7				

Notes: C = personal consumption expenditure.
I = gross private domestic investment.
Y = gross domestic product.

Source: *Economic Report of the President,* 2008, Table B-2.

gallon); quantgas (thousands of barrels per day, unleaded); persincome (personal income, billions of dollars); and car sales (millions of cars per year).

a. Develop a suitable supply-and-demand model for gasoline consumption.

b. Which variables in the model in (*a*) are endogenous and which are exogenous?

c. If you estimate the demand-and-supply functions that you have developed by OLS, will your results be reliable? Why or why not?

d. Save the OLS estimates of your demand-and-supply functions for another look after we discuss Chapter 20.

18.14. Table 18.4, found on the textbook website, gives macroeconomic data on several variables for the U.S. economy for the quarterly periods 1951–I to 2000–IV.* The variables are as follows: *Year* = date; *Qtr* = quarter; *Realgdp* = real GDP (billions of dollars); *Realcons* = real consumption expenditure; *Realinvs* = real investment by private sector; *Realgovt* = real government expenditure; *Realdpi* = real disposable personal income; *CPI_U* = consumer price index; *M*1 = nominal money stock; *Tbilrate* = quarterly average of month-end 90-day T-bill rate; *Pop* = population, millions, interpolate of year-end figures using constant growth rate per quarter; *Infl* = rate of inflation (first observation is missing); and *Realint* = expost real interest rate = *Tbilrate–Infl* (first observation missing).

Using these data, develop a simple macroeconomic model of the U.S. economy. You will be asked to estimate this model in Chapter 20.

*These data are originally from the Department of Commerce, Bureau of Economic Analysis, and from www.economagic.com, and are reproduced from William H. Greene, *Econometric Analysis,* 6th ed., 2008, Table F5.1, p.1083.

Chapter 19

The Identification Problem

In this chapter we consider the nature and significance of the identification problem. The crux of the identification problem is as follows: Recall the demand-and-supply model introduced in Section 18.2. Suppose that we have time series data on Q and P only and no additional information (such as income of the consumer, price prevailing in the previous period, and weather condition). The identification problem then consists in seeking an answer to this question: Given only the data on P and Q, how do we know whether we are estimating the demand function or the supply function? Alternatively, if we *think* we are fitting a demand function, how do we guarantee that it is, in fact, the demand function that we are estimating and not something else?

A moment's reflection will reveal that an answer to the preceding question is necessary before one proceeds to estimate the parameters of our demand function. In this chapter we shall show how the identification problem is resolved. We first introduce a few notations and definitions and then illustrate the identification problem with several examples. This is followed by the rules that may be used to find out whether an equation in a simultaneous-equation model is identified, that is, whether it is the relationship that we are actually estimating, be it the demand or supply function or something else.

19.1 Notations and Definitions

To facilitate our discussion, we introduce the following notations and definitions.

The general M equations model in M endogenous, or jointly dependent, variables may be written as Eq. (19.1.1):

$$\begin{aligned}
Y_{1t} &= \qquad\qquad\quad \beta_{12}Y_{2t} + \beta_{13}Y_{3t} + \cdots + \beta_{1M}Y_{Mt} \\
&\qquad\qquad + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + \cdots + \gamma_{1K}X_{Kt} + u_{1t} \\
Y_{2t} &= \beta_{21}Y_{1t} \qquad\quad + \beta_{23}Y_{3t} + \cdots + \beta_{2M}Y_{Mt} \\
&\qquad\qquad + \gamma_{21}X_{1t} + \gamma_{22}X_{2t} + \cdots + \gamma_{2K}X_{Kt} + u_{2t} \\
Y_{3t} &= \beta_{31}Y_{1t} + \beta_{32}Y_{2t} \qquad\quad + \cdots + \beta_{3M}Y_{Mt} \\
&\qquad\qquad + \gamma_{31}X_{1t} + \gamma_{32}X_{2t} + \cdots + \gamma_{3K}X_{Kt} + u_{3t} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
Y_{MT} &= \beta_{M1}Y_{1t} + \beta_{M2}Y_{2t} + \cdots + \beta_{M,M-1}Y_{M-1,t} \\
&\qquad\qquad + \gamma_{M1}X_{1t} + \gamma_{M2}X_{2t} + \cdots + \gamma_{MK}X_{Kt} + u_{Mt}
\end{aligned} \tag{19.1.1}$$

where $Y_1, Y_2, \ldots, Y_M = M$ endogenous, or jointly dependent, variables
$X_1, X_2, \ldots, X_K = K$ predetermined variables (one of these X variables may take a value of unity to allow for the intercept term in each equation)
$u_1, u_2, \ldots, u_M = M$ stochastic disturbances
$t = 1, 2, \ldots, T =$ total number of observations
β's = coefficients of the endogenous variables
γ's = coefficients of the predetermined variables

In passing, note that not each and every variable need appear in each equation. As a matter of fact, we see in Section 19.2 that this must not be the case if an equation is to be identified.

As Eq. (19.1.1) shows, the variables entering a simultaneous-equation model are of two types: **endogenous,** that is, those (whose values are) determined within the model; and **predetermined,** that is, those (whose values are) determined outside the model. The endogenous variables are regarded as stochastic, whereas the predetermined variables are treated as nonstochastic.

The predetermined variables are divided into two categories: **exogenous,** current as well as lagged, and **lagged endogenous.** Thus, X_{1t} is a current (present-time) exogenous variable, whereas $X_{1(t-1)}$ is a lagged exogenous variable, with a lag of one time period. $Y_{(t-1)}$ is a lagged endogenous variable with a lag of one time period, but since the value of $Y_{1(t-1)}$ is known at the current time t, it is regarded as nonstochastic, hence, a predetermined variable.[1] In short, current exogenous, lagged exogenous, and lagged endogenous variables are deemed predetermined; their values are not determined by the model in the current time period.

It is up to the model builder to specify which variables are endogenous and which are predetermined. Although (noneconomic) variables, such as temperature and rainfall, are clearly exogenous or predetermined, the model builder must exercise great care in classifying economic variables as endogenous or predetermined: He or she must defend the classification on a priori or theoretical grounds. However, later in the chapter we provide a statistical test of exogeneity.

The equations appearing in (19.1.1) are known as the **structural,** or **behavioral,** equations because they may portray the structure (of an economic model) of an economy or the behavior of an economic agent (e.g., consumer or producer). The β's and γ's are known as the **structural parameters** or **coefficients.**

From the structural equations one can solve for the M endogenous variables and derive the **reduced-form equations** and the associated **reduced-form coefficients. A reduced-form equation is one that expresses an endogenous variable solely in terms of the predetermined variables and the stochastic disturbances.** To illustrate, consider the Keynesian model of income determination encountered in Chapter 18:

Consumption function: $$C_t = \beta_0 + \beta_1 Y_t + u_t \qquad 0 < \beta_1 < 1 \tag{18.2.3}$$

Income identity: $$Y_t = C_t + I_t \tag{18.2.4}$$

In this model C (consumption) and Y (income) are the endogenous variables and I (investment expenditure) is treated as an exogenous variable. Both these equations are structural equations, Eq. (18.2.4) being an identity. As usual, the MPC β_1 is assumed to lie between 0 and 1.

If Eq. (18.2.3) is substituted into Eq. (18.2.4), we obtain, after simple algebraic manipulation,

$$Y_t = \Pi_0 + \Pi_1 I_t + w_t \tag{19.1.2}$$

[1]It is assumed implicitly here that the stochastic disturbances, the u's, are serially uncorrelated. If this is not the case, Y_{t-1} will be correlated with the current period disturbance term u_t. Hence, we cannot treat it as predetermined.

where

$$\Pi_0 = \frac{\beta_0}{1 - \beta_1}$$
$$\Pi_1 = \frac{1}{1 - \beta_1} \qquad \textbf{(19.1.3)}$$
$$w_t = \frac{u_t}{1 - \beta_1}$$

Equation (19.1.2) is a **reduced-form equation;** it expresses the endogenous variable Y solely as a function of the exogenous (or predetermined) variable I and the stochastic disturbance term u. Π_0 and Π_1 are the associated **reduced-form coefficients.** Notice that these reduced-form coefficients are nonlinear combinations of the structural coefficient(s).

Substituting the value of Y from Eq. (19.1.2) into C of Eq. (18.2.3), we obtain another reduced-form equation:

$$C_t = \Pi_2 + \Pi_3 I_t + w_t \qquad \textbf{(19.1.4)}$$

where

$$\Pi_2 = \frac{\beta_0}{1 - \beta_1} \qquad \Pi_3 = \frac{\beta_1}{1 - \beta_1}$$
$$w_t = \frac{u_t}{1 - \beta_1} \qquad \textbf{(19.1.5)}$$

The reduced-form coefficients, such as Π_1 and Π_3, are also known as **impact,** or **short-run, multipliers,** because they measure the immediate impact on the endogenous variable of a unit change in the value of the exogenous variable.[2] If in the preceding Keynesian model the investment expenditure is increased by, say, \$1 and if the MPC is assumed to be 0.8, then from Eq. (19.1.3) we obtain $\Pi_1 = 5$. This result means that increasing the investment by \$1 will immediately (i.e., in the current time period) lead to an increase in income of \$5, that is, a fivefold increase. Similarly, under the assumed conditions, Eq. (19.1.5) shows that $\Pi_3 = 4$, meaning that \$1 increase in investment expenditure will lead immediately to \$4 increase in consumption expenditure.

In the context of econometric models, equations such as Eq. (18.2.4) or $Q_t^d = Q_t^s$ (quantity demanded equal to quantity supplied) are known as the *equilibrium conditions*. Identity (18.2.4) states that aggregate income Y must be equal to aggregate consumption (i.e., consumption expenditure plus investment expenditure). When equilibrium is achieved, the endogenous variables assume their equilibrium values.[3]

Notice an interesting feature of the reduced-form equations. Since only the predetermined variables and stochastic disturbances appear on the right sides of these equations, and since the predetermined variables are assumed to be uncorrelated with the disturbance terms, the OLS method can be applied to estimate the coefficients of the reduced-form equations (the Π's). From the estimated reduced-form coefficients one may estimate the structural coefficients (the β's), as shown later. This procedure is known as **indirect least squares** (ILS), and the estimated structural coefficients are called ILS estimates.

[2]In econometric models the exogenous variables play a crucial role. Very often, such variables are under the direct control of the government. Examples are the rate of personal and corporate taxes, subsidies, unemployment compensation, etc.

[3]For details, see Jan Kmenta, *Elements of Econometrics,* 2d ed., Macmillan, New York, 1986, pp. 723–731.

We shall study the ILS method in greater detail in Chapter 20. In the meantime, note that since the reduced-form coefficients can be estimated by the OLS method, and since these coefficients are combinations of the structural coefficients, the possibility exists that the structural coefficients can be "retrieved" from the reduced-form coefficients, and it is in the estimation of the structural parameters that we may be ultimately interested. How does one retrieve the structural coefficients from the reduced-form coefficients? The answer is given in Section 19.2, an answer that brings out the crux of the identification problem.

19.2 The Identification Problem

By the **identification problem** we mean whether numerical estimates of the parameters of a structural equation can be obtained from the estimated reduced-form coefficients. If this can be done, we say that the particular equation is *identified.* If this cannot be done, then we say that the equation under consideration is *unidentified,* or *underidentified.*

An identified equation may be either *exactly* (or fully or just) *identified* or *overidentified*. It is said to be exactly identified if unique numerical values of the structural parameters can be obtained. It is said to be overidentified if more than one numerical value can be obtained for some of the parameters of the structural equations. The circumstances under which each of these cases occurs will be shown in the following discussion.

The identification problem arises because different sets of structural coefficients may be compatible with the same set of data. To put the matter differently, a given reduced-form equation may be compatible with different structural equations or different hypotheses (models), and it may be difficult to tell which particular hypothesis (model) we are investigating. In the remainder of this section we consider several examples to show the nature of the identification problem.

Underidentification

Consider once again the demand-and-supply model (18.2.1) and (18.2.2), together with the market-clearing, or equilibrium, condition that demand is equal to supply. By the equilibrium condition, we obtain

$$\alpha_0 + \alpha_1 P_t + u_{1t} = \beta_0 + \beta_1 P_t + u_{2t} \tag{19.2.1}$$

Solving Eq. (19.2.1), we obtain the equilibrium price

$$P_t = \Pi_0 + v_t \tag{19.2.2}$$

where

$$\Pi_0 = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \tag{19.2.3}$$

$$v_t = \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1} \tag{19.2.4}$$

Substituting P_t from Eq. (19.2.2) into Eq. (18.2.1) or (18.2.2), we obtain the following equilibrium quantity:

$$Q_t = \Pi_1 + w_t \tag{19.2.5}$$

where

$$\Pi_1 = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} \tag{19.2.6}$$

$$w_t = \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1} \tag{19.2.7}$$

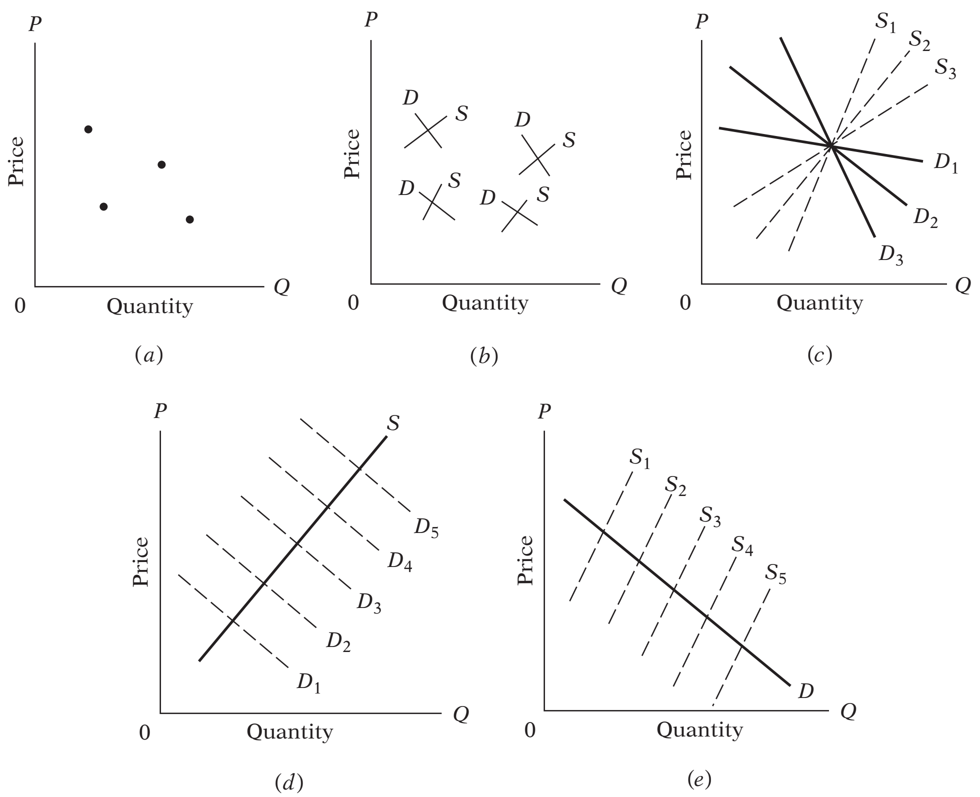
Incidentally, note that the error terms v_t and w_t are linear combinations of the original error terms u_1 and u_2.

Equations (19.2.2) and (19.2.5) are reduced-form equations. Now our demand-and-supply model contains four structural coefficients α_0, α_1, β_0, and β_1, but there is no unique way of estimating them. Why? The answer lies in the two reduced-form coefficients given in Eqs. (19.2.3) and (19.2.6). These reduced-form coefficients contain all four structural parameters, but there is no way in which the four structural unknowns can be estimated from only two reduced-form coefficients. Recall from high school algebra that to estimate four unknowns we must have four (independent) equations, and, in general, to estimate k unknowns we must have k (independent) equations. Incidentally, if we run the reduced-form regression (19.2.2) and (19.2.5), we will see that there are no explanatory variables, only the *constants,* and these *constants* will simply give the mean values of P and Q (why?).

What all this means is that, given time series data on P (price) and Q (quantity) and no other information, there is no way the researcher can guarantee whether he or she is estimating the demand function or the supply function. That is, a given P_t and Q_t represent simply the point of intersection of the appropriate demand-and-supply curves because of the equilibrium condition that demand is equal to supply. To see this clearly, consider the scattergram shown in Figure 19.1.

Figure 19.1*a* gives a few scatterpoints relating Q to P. Each scatterpoint represents the intersection of a demand and a supply curve, as shown in Figure 19.1*b*. Now consider a single point, such as that shown in Figure 19.1*c*. There is no way we can be sure which demand-and-supply curve of a whole family of curves shown in that panel generated that point. Clearly, some additional information about the nature of the demand-and-supply curves is needed. For example, if the demand curve shifts over time because of change in income,

FIGURE 19.1 Hypothetical supply-and-demand functions and the identification problem.



tastes, etc., but the supply curve remains relatively stable, as in Figure 19.1*d*, the scatterpoints trace out a supply curve. In this situation, we say that the supply curve is identified. By the same token, if the supply curve shifts over time because of changes in weather conditions (in the case of agricultural commodities) or other extraneous factors but the demand curve remains relatively stable, as in Figure 19.1*e*, the scatterpoints trace out a demand curve. In this case, we say that the demand curve is identified.

There is an alternative and perhaps more illuminating way of looking at the identification problem. Suppose we multiply Eq. (18.2.1) by λ $(0 \le \lambda \le 1)$ and Eq. (18.2.2) by $1 - \lambda$ to obtain the following equations (*note:* we drop the superscripts on Q):

$$\lambda Q_t = \lambda\alpha_0 + \lambda\alpha_1 P_t + \lambda u_{1t} \tag{19.2.8}$$

$$(1-\lambda)Q_t = (1-\lambda)\beta_0 + (1-\lambda)\beta_1 P_t + (1-\lambda)u_{2t} \tag{19.2.9}$$

Adding these two equations gives the following *linear combination* of the original demand-and-supply equations:

$$Q_t = \gamma_0 + \gamma_1 P_t + w_t \tag{19.2.10}$$

where

$$\begin{aligned} \gamma_0 &= \lambda\alpha_0 + (1-\lambda)\beta_0 \\ \gamma_1 &= \lambda\alpha_1 + (1-\lambda)\beta_1 \\ w_t &= \lambda u_{1t} + (1-\lambda)u_{2t} \end{aligned} \tag{19.2.11}$$

The "bogus," or "mongrel," equation (19.2.10) is *observationally indistinguishable* from either Eq. (18.2.1) or Eq. (18.2.2) because they involve the regression of Q and P. Therefore, if we have time series data on P and Q only, any of Eqs. (18.2.1), (18.2.2), or (19.2.10) may be compatible with the same data. In other words, the same data may be compatible with the "hypothesis" Eqs. (18.2.1), (18.2.2), or (19.2.10), and there is no way we can tell which one of these hypotheses we are testing.

For an equation to be identified, that is, for its parameters to be estimated, it must be shown that the given set of data will not produce a structural equation that looks similar in appearance to the one in which we are interested. If we set out to estimate the demand function, we must show that the given data are not consistent with the supply function or some mongrel equation.

Just, or Exact, Identification

The reason we could not identify the preceding demand function or the supply function was that the same variables P and Q are present in both functions and there is no additional information, such as that indicated in Figure 19.1*d* or *e*. But suppose we consider the following demand-and-supply model:

Demand function: $$Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \qquad \alpha_1 < 0, \alpha_2 > 0 \tag{19.2.12}$$

Supply function: $$Q_t = \beta_0 + \beta_1 P_t + u_{2t} \qquad \beta_1 > 0 \tag{19.2.13}$$

where I = income of the consumer, an exogenous variable, and all other variables are as defined previously.

Notice that the only difference between the preceding model and our original demand-and-supply model is that there is an additional variable in the demand function, namely, income. From economic theory of demand we know that income is usually an important determinant of demand for most goods and services. Therefore, its inclusion in the demand function will give us some additional information about consumer behavior. For most commodities income is expected to have a positive effect on consumption $(\alpha_2 > 0)$.

Using the market-clearing mechanism, quantity demanded = quantity supplied, we have

$$\alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} = \beta_0 + \beta_1 P_t + u_{2t} \tag{19.2.14}$$

Solving Eq. (19.2.14) provides the following equilibrium value of P_t:

$$P_t = \Pi_0 + \Pi_1 I_t + v_t \tag{19.2.15}$$

where the reduced-form coefficients are

$$\begin{aligned} \Pi_0 &= \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \\ \Pi_1 &= -\frac{\alpha_2}{\alpha_1 - \beta_1} \end{aligned} \tag{19.2.16}$$

and

$$v_t = \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1}$$

Substituting the equilibrium value of P_t into the preceding demand or supply function, we obtain the following equilibrium quantity:

$$Q_t = \Pi_2 + \Pi_3 I_t + w_t \tag{19.2.17}$$

where

$$\begin{aligned} \Pi_2 &= \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} \\ \Pi_3 &= -\frac{\alpha_2\beta_1}{\alpha_1 - \beta_1} \end{aligned} \tag{19.2.18}$$

and

$$w_t = \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1}$$

Since Eqs. (19.2.15) and (19.2.17) are both reduced-form equations, the ordinary least squares (OLS) method can be applied to estimate their parameters. Now the demand-and-supply model (19.2.12) and (19.2.13) contains five structural coefficients—α_0, α_1, α_2, β_0 and β_1. But there are only four equations to estimate them, namely, the four reduced-form coefficients Π_0, Π_1, Π_2, and Π_3 given in Eqs. (19.2.16) and (19.2.18). Hence, unique solution of all the structural coefficients is not possible. But it can be readily shown that the parameters of the supply function can be identified (estimated) because

$$\begin{aligned} \beta_0 &= \Pi_2 - \beta_1\Pi_0 \\ \beta_1 &= \frac{\Pi_3}{\Pi_1} \end{aligned} \tag{19.2.19}$$

But there is no unique way of estimating the parameters of the demand function; therefore, it remains underidentified. Incidentally, note that the structural coefficient β_1 is a nonlinear function of the reduced-form coefficients, which poses some problems when it comes to estimating the standard error of the estimated β_1, as we shall see in Chapter 20.

To verify that the demand function (19.2.12) cannot be identified (estimated), let us multiply it by λ $(0 \le \lambda \le 1)$ and (19.2.13) by $1 - \lambda$ and add them up to obtain the following "mongrel" equation:

$$Q_t = \gamma_0 + \gamma_1 P_t + \gamma_2 I_t + w_t \tag{19.2.20}$$

where

$$\begin{aligned} \gamma_0 &= \lambda\alpha_0 + (1-\lambda)\beta_0 \\ \gamma_1 &= \lambda\alpha_1 + (1-\lambda)\beta_1 \\ \gamma_2 &= \lambda\alpha_2 \end{aligned} \tag{19.2.21}$$

and

$$w_t = \lambda u_{1t} + (1-\lambda)u_{2t}$$

Equation (19.2.20) is observationally indistinguishable from the demand function (19.2.12) although it is distinguishable from the supply function (19.2.13), which does not contain the variable I as an explanatory variable. Hence, the demand function remains unidentified.

Notice an interesting fact: It is the presence of an additional variable in the demand function that enables us to identify the supply function! Why? The inclusion of the income variable in the demand equation provides us some additional information about the variability of the function, as indicated in Figure 19.1*d*. The figure shows how the intersection of the stable supply curve with the shifting demand curve (on account of changes in income) enables us to trace (identify) the supply curve. As will be shown shortly, very often the identifiability of an equation depends on whether it excludes one or more variables that are included in other equations in the model.

But suppose we consider the following demand-and-supply model:

$$\textit{Demand function:}\quad Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \qquad \alpha_1 < 0, \alpha_2 > 0 \tag{19.2.12}$$

$$\textit{Supply function:}\quad Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \qquad \beta_1 > 0, \beta_2 > 0 \tag{19.2.22}$$

where the demand function remains as before but the supply function includes an additional explanatory variable, price lagged one period. The supply function postulates that the quantity of a commodity supplied depends on its current and previous period's price, a model often used to explain the supply of many agricultural commodities. Note that P_{t-1} is a predetermined variable because its value is known at time t.

By the market-clearing mechanism we have

$$\alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \tag{19.2.23}$$

Solving this equation, we obtain the following equilibrium price:

$$P_t = \Pi_0 + \Pi_1 I_t + \Pi_2 P_{t-1} + v_t \tag{19.2.24}$$

where

$$\begin{aligned} \Pi_0 &= \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \\ \Pi_1 &= -\frac{\alpha_2}{\alpha_1 - \beta_1} \\ \Pi_2 &= \frac{\beta_2}{\alpha_1 - \beta_1} \\ v_t &= \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1} \end{aligned} \tag{19.2.25}$$

Substituting the equilibrium price into the demand or supply equation, we obtain the corresponding equilibrium quantity:

$$Q_t = \Pi_3 + \Pi_4 I_t + \Pi_5 P_{t-1} + w_t \tag{19.2.26}$$

where the reduced-form coefficients are

$$\Pi_3 = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1}$$
$$\Pi_4 = -\frac{\alpha_2\beta_1}{\alpha_1 - \beta_1} \tag{19.2.27}$$
$$\Pi_5 = \frac{\alpha_1\beta_2}{\alpha_1 - \beta_1}$$

and

$$w_t = \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1}$$

The demand-and-supply model given in Eqs. (19.2.12) and (19.2.22) contains six structural coefficients—$\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$, and β_2—and there are six reduced-form coefficients—$\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4$, and Π_5—to estimate them. Thus, we have six equations in six unknowns, and normally we should be able to obtain unique estimates. Therefore, the parameters of both the demand-and-supply equations can be identified, and the system as a whole can be identified. (In Exercise 19.2 the reader is asked to express the six structural coefficients in terms of the six reduced-form coefficients given previously to show that unique estimation of the model is possible.)

To check that the preceding demand-and-supply functions are identified, we can also resort to the device of multiplying the demand equation (19.2.12) by λ ($0 \le \lambda \le 1$) and the supply equation (19.2.22) by $1 - \lambda$ and add them to obtain a mongrel equation. This mongrel equation will contain both the predetermined variables I_t and P_{t-1}; hence, it will be observationally different from the demand as well as the supply equation because the former does not contain P_{t-1} and the latter does not contain I_t.

Overidentification

For certain goods and services, income as well as wealth of the consumer is an important determinant of demand. Therefore, let us modify the demand function (19.2.12) as follows, keeping the supply function as before:

Demand function: $$Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{1t} \tag{19.2.28}$$

Supply function: $$Q_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \tag{19.2.22}$$

where in addition to the variables already defined, R represents wealth; for most goods and services, wealth, like income, is expected to have a positive effect on consumption.

Equating demand to supply, we obtain the following equilibrium price and quantity:

$$P_t = \Pi_0 + \Pi_1 I_t + \Pi_2 R_t + \Pi_3 P_{t-1} + v_t \tag{19.2.29}$$

$$Q_t = \Pi_4 + \Pi_5 I_t + \Pi_6 R_t + \Pi_7 P_{t-1} + w_t \tag{19.2.30}$$

where

$$\Pi_0 = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \qquad \Pi_1 = -\frac{\alpha_2}{\alpha_1 - \beta_1}$$
$$\Pi_2 = -\frac{\alpha_3}{\alpha_1 - \beta_1} \qquad \Pi_3 = \frac{\beta_2}{\alpha_1 - \beta_1}$$
$$\Pi_4 = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} \qquad \Pi_5 = -\frac{\alpha_2\beta_1}{\alpha_1 - \beta_1} \tag{19.2.31}$$
$$\Pi_6 = -\frac{\alpha_3\beta_1}{\alpha_1 - \beta_1} \qquad \Pi_7 = \frac{\alpha_1\beta_2}{\alpha_1 - \beta_1}$$
$$w_t = \frac{\alpha_1 u_{2t} - \beta_1 u_{1t}}{\alpha_1 - \beta_1} \qquad v_t = \frac{u_{2t} - u_{1t}}{\alpha_1 - \beta_1}$$

The preceding demand-and-supply model contains seven structural coefficients, but there are eight equations to estimate them—the eight reduced-form coefficients given in Eq. (19.2.31); that is, the number of equations is greater than the number of unknowns. As a result, unique estimation of all the parameters of our model is not possible, which can be shown easily. From the preceding reduced-form coefficients, we can obtain

$$\beta_1 = \frac{\Pi_6}{\Pi_2} \tag{19.2.32}$$

or

$$\beta_1 = \frac{\Pi_5}{\Pi_1} \tag{19.2.33}$$

that is, there are two estimates of the price coefficient in the supply function, and there is no guarantee that these two values or solutions will be identical.[4] Moreover, since β_1 appears in the denominators of all the reduced-form coefficients, the ambiguity in the estimation of β_1 will be transmitted to other estimates too.

Why was the supply function identified in the system (19.2.12) and (19.2.22) but not in the system (19.2.28) and (19.2.22), although in both cases the supply function remains the same? The answer is that we have "too much," or an **oversufficiency of information,** to identify the supply curve. This situation is the opposite of the case of underidentification, where there is too little information. The oversufficiency of the information results from the fact that in the model (19.2.12) and (19.2.22) the exclusion of the income variable from the supply function was enough to identify it, but in the model (19.2.28) and (19.2.22) the supply function excludes not only the income variable but also the wealth variable. In other words, in the latter model we put "too many" restrictions on the supply function by requiring it to exclude more variables than necessary to identify it. However, this situation does not imply that overidentification is necessarily bad because we shall see in Chapter 20 how we can handle the problem of too much information, or too many restrictions.

We have now exhausted all the cases. As the preceding discussion shows, an equation in a simultaneous-equation model may be underidentified or identified (either over- or just). The model as a whole is identified if each equation in it is identified. To secure identification, we resort to the reduced-form equations. But in Section 19.3, we consider an alternative and perhaps less time-consuming method of determining whether or not an equation in a simultaneous-equation model is identified.

[4]Notice the difference between under- and overidentification. In the former case, it is impossible to obtain estimates of the structural parameters, whereas in the latter case, there may be several estimates of one or more structural coefficients.

19.3 Rules for Identification

As the examples in Section 19.2 show, in principle it is possible to resort to the reduced-form equations to determine the identification of an equation in a system of simultaneous equations. But these examples also show how time-consuming and laborious the process can be. Fortunately, it is not essential to use this procedure. The so-called **order and rank conditions of identification** lighten the task by providing a systematic routine.

To understand the order and rank conditions, we introduce the following notations:

M = number of endogenous variables in the model
m = number of endogenous variables in a given equation
K = number of predetermined variables in the model including the intercept
k = number of predetermined variables in a given equation

The Order Condition of Identifiability[5]

A necessary (but not sufficient) condition of identification, known as the **order condition,** may be stated in two different but equivalent ways as follows (the necessary as well as sufficient condition of identification will be presented shortly):

Definition 19.1 In a model of M simultaneous equations, in order for an equation to be identified, it must exclude *at least* $M - 1$ variables (endogenous as well as predetermined) appearing in the model. If it excludes exactly $M - 1$ variables, the equation is just identified. If it excludes more than $M - 1$ variables, it is overidentified.

Definition 19.2 In a model of M simultaneous equations, in order for an equation to be identified, the number of predetermined variables excluded from the equation must not be less than the number of endogenous variables included in that equation less 1, that is,

$$K - k \geq m - 1 \tag{19.3.1}$$

If $K - k = m - 1$, the equation is just identified, but if $K - k > m - 1$, it is overidentified.

In Exercise 19.1 the reader is asked to prove that the preceding two definitions of identification are equivalent.

To illustrate the order condition, let us revert to our previous examples.

EXAMPLE 19.1

$$\text{Demand function:} \quad Q_t^d = \alpha_0 + \alpha_1 P_t + u_{1t} \tag{18.2.1}$$

$$\text{Supply function:} \quad Q_t^s = \beta_0 + \beta_1 P_t + u_{2t} \tag{18.2.2}$$

This model has two endogenous variables P and Q and no predetermined variables. To be identified, each of these equations must exclude at least $M - 1 = 1$ variable. Since this is not the case, neither equation is identified.

EXAMPLE 19.2

$$\text{Demand function:} \quad Q_t^d = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \tag{19.2.12}$$

$$\text{Supply function:} \quad Q_t^s = \beta_0 + \beta_1 P_t + u_{2t} \tag{19.2.13}$$

In this model Q and P are endogenous and I is exogenous. Applying the order condition given in Eq. (19.3.1), we see that the demand function is unidentified. On the other hand, the supply function is just identified because it excludes exactly $M - 1 = 1$ variable I_t.

[5]The term **order** refers to the order of a matrix, that is, the number of rows and columns present in a matrix. See **Appendix B.**

EXAMPLE 19.3

$$\text{Demand function:} \quad Q_t^d = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + u_{1t} \tag{19.2.12}$$

$$\text{Supply function:} \quad Q_t^s = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \tag{19.2.22}$$

Given that P_t and Q_t are endogenous and I_t and P_{t-1} are predetermined, Eq. (19.2.12) excludes exactly one variable P_{t-1} and Eq. (19.2.22) also excludes exactly one variable I_t. Hence each equation is identified by the order condition. Therefore, the model as a whole is identified.

EXAMPLE 19.4

$$\text{Demand function:} \quad Q_t^d = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{1t} \tag{19.2.28}$$

$$\text{Supply function:} \quad Q_t^s = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + u_{2t} \tag{19.2.22}$$

In this model P_t and Q_t are endogenous and I_t, R_t, and P_{t-1} are predetermined. The demand function excludes exactly one variable P_{t-1}, and hence by the order condition it is exactly identified. But the supply function excludes two variables I_t and R_t, and hence it is overidentified. As noted before, in this case there are two ways of estimating β_1, the coefficient of the price variable.

Notice a slight complication here. By the order condition the demand function is identified. But if we try to estimate the parameters of this equation from the reduced-form coefficients given in Eq. (19.2.31), the estimates will not be unique because β_1, which enters into the computations, takes two values and we shall have to decide which of these values is appropriate. But this complication can be obviated because it is shown in Chapter 20 that in cases of overidentification the method of indirect least squares is not appropriate and should be discarded in favor of other methods. One such method is **two-stage least squares,** which we shall discuss fully in Chapter 20.

As the previous examples show, identification of an equation in a model of simultaneous equations is possible if that equation excludes one or more variables that are present elsewhere in the model. This situation is known as the *exclusion* (of variables) *criterion,* or the *zero restrictions criterion* (the coefficients of variables not appearing in an equation are assumed to have zero values). This criterion is by far the most commonly used method of securing or determining identification of an equation. But notice that the zero restrictions criterion is based on a priori or theoretical expectations that certain variables do not appear in a given equation. It is up to the researcher to spell out clearly why he or she does expect certain variables to appear in some equations and not in others.

The Rank Condition of Identifiability[6]

The order condition discussed previously is *a necessary but not sufficient* condition for identification; that is, even if it is satisfied, it may happen that an equation is not identified. Thus, in Example 19.2, the supply equation was identified by the order condition because it excluded the income variable I_t, which appeared in the demand function. But identification is accomplished only if α_2, the coefficient of I_t in the demand function, is not zero, that is, if the income variable not only probably but actually does enter the demand function.

More generally, even if the order condition $K - k \geq m - 1$ is satisfied by an equation, it may be unidentified because the predetermined variables excluded from this equation but present in the model may not all be independent so that there may not be one-to-one correspondence between the structural coefficients (the β's) and the reduced-form coefficients

[6]The term **rank** refers to the rank of a matrix and is given by the largest-order square matrix (contained in the given matrix) whose determinant is nonzero. Alternatively, the rank of a matrix is the largest number of linearly independent rows or columns of that matrix. See **Appendix B.**

(the Π's). That is, we may not be able to estimate the structural parameters from the reduced-form coefficients, as we shall show shortly. Therefore, we need both a necessary and sufficient condition for identification. This is provided by the *rank condition* of identification, which may be stated as follows:

Rank Condition of Identification

In a model containing M equations in M endogenous variables, an equation is identified if and only if *at least* one nonzero determinant of order $(M-1)(M-1)$ can be constructed from the coefficients of the variables (both endogenous and predetermined) excluded from that particular equation but included in the other equations of the model.

As an illustration of the rank condition of identification, consider the following hypothetical system of simultaneous equations in which the Y variables are endogenous and the X variables are predetermined.[7]

$$Y_{1t} - \beta_{10} - \beta_{12}Y_{2t} - \beta_{13}Y_{3t} - \gamma_{11}X_{1t} = u_{1t} \quad \textbf{(19.3.2)}$$

$$Y_{2t} - \beta_{20} - \beta_{23}Y_{3t} - \gamma_{21}X_{1t} - \gamma_{22}X_{2t} = u_{2t} \quad \textbf{(19.3.3)}$$

$$Y_{3t} - \beta_{30} - \beta_{31}Y_{1t} - \gamma_{31}X_{1t} - \gamma_{32}X_{2t} = u_{3t} \quad \textbf{(19.3.4)}$$

$$Y_{4t} - \beta_{40} - \beta_{41}Y_{1t} - \beta_{42}Y_{2t} - \gamma_{43}X_{3t} = u_{4t} \quad \textbf{(19.3.5)}$$

To facilitate identification, let us write the preceding system in Table 19.1, which is self-explanatory.

Let us first apply the order condition of identification, as shown in Table 19.2. By the order condition each equation is identified. Let us recheck with the rank condition. Consider the first equation, which excludes variables Y_4, X_2, and X_3 (this is represented by zeros in the first row of Table 19.1). For this equation to be identified, we must obtain at

TABLE 19.1

	Coefficients of the Variables							
Equation No.	1	Y_1	Y_2	Y_3	Y_4	X_1	X_2	X_3
(19.3.2)	$-\beta_{10}$	1	$-\beta_{12}$	$-\beta_{13}$	0	$-\gamma_{11}$	0	0
(19.3.3)	$-\beta_{20}$	0	1	$-\beta_{23}$	0	$-\gamma_{21}$	$-\gamma_{22}$	0
(19.3.4)	$-\beta_{30}$	$-\beta_{31}$	0	1	0	$-\gamma_{31}$	$-\gamma_{32}$	0
(19.3.5)	$-\beta_{40}$	$-\beta_{41}$	$-\beta_{42}$	0	1	0	0	$-\gamma_{43}$

TABLE 19.2

Equation No.	No. of Predetermined Variables Excluded, $(K-k)$	No. of Endogenous Variables Included, Less One, $(m-1)$	Identified?
(19.3.2)	2	2	Exactly
(19.3.3)	1	1	Exactly
(19.3.4)	1	1	Exactly
(19.3.5)	2	2	Exactly

[7]The simultaneous-equation system presented in Eq. (19.1.1) may be shown in the following alternative form, which may be convenient for matrix manipulations.

least one nonzero determinant of order 3 × 3 from the coefficients of the variables excluded from this equation but included in other equations. To obtain the determinant we first obtain the relevant matrix of coefficients of variables Y_4, X_2, and X_3 included in the other equations. In the present case there is only one such matrix, call it **A**, defined as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & -\gamma_{22} & 0 \\ 0 & -\gamma_{32} & 0 \\ 1 & 0 & -\gamma_{43} \end{bmatrix} \tag{19.3.6}$$

It can be seen that the determinant of this matrix is zero:

$$\det \mathbf{A} = \begin{vmatrix} 0 & -\gamma_{22} & 0 \\ 0 & -\gamma_{32} & 0 \\ 1 & 0 & -\gamma_{43} \end{vmatrix} \tag{19.3.7}$$

Since the determinant is zero, the rank of the matrix (19.3.6), denoted by $\rho(\mathbf{A})$, is less than 3. Therefore, Eq. (19.3.2) does not satisfy the rank condition and hence is not identified.

As noted, the rank condition is both a necessary and sufficient condition for identification. Therefore, although the order condition shows that Eq. (19.3.2) is identified, the rank condition shows that it is not. Apparently, the columns or rows of the matrix **A** given in Eq. (19.3.6) are not (linearly) independent, meaning that there is some relationship between the variables Y_4, X_2, and X_3. As a result, we may not have enough information to estimate the parameters of equation (19.3.2); the reduced-form equations for the preceding model will show that it is not possible to obtain the structural coefficients of that equation from the reduced-form coefficients. The reader should verify that by the rank condition Eqs. (19.3.3) and (19.3.4) are also unidentified but Eq. (19.3.5) is identified.

As the preceding discussion shows, *the rank condition tells us whether the equation under consideration is identified or not, whereas the order condition tells us if it is exactly identified or overidentified.*

To apply the rank condition one may proceed as follows:

1. Write down the system in a tabular form, as shown in Table 19.1.
2. Strike out the coefficients of the row in which the equation under consideration appears.
3. Also strike out the columns corresponding to those coefficients in step (2) which are nonzero.
4. The entries left in the table will then give only the coefficients of the variables included in the system but not in the equation under consideration. From these entries form all possible matrices, like **A**, of order $M - 1$ and obtain the corresponding determinants. If at least one nonvanishing or nonzero determinant can be found, the equation in question is (just or over-) identified. The rank of the matrix, say, **A**, in this case is exactly equal to $M - 1$. If all the possible $(M - 1)(M - 1)$ determinants are zero, the rank of the matrix **A** is less than $M - 1$ and the equation under investigation is not identified.

Our discussion of the order and rank conditions of identification leads to the following general principles of identifiability of a structural equation in a system of M simultaneous equations:

1. If $K - k > m - 1$ and the rank of the **A** matrix is $M - 1$, the equation is overidentified.
2. If $K - k = m - 1$ and the rank of the matrix **A** is $M - 1$, the equation is exactly identified.
3. If $K - k \geq m - 1$ and the rank of the matrix **A** is less than $M - 1$, the equation is underidentified.
4. If $K - k < m - 1$, the structural equation is unidentified. The rank of the **A** matrix in this case is bound to be less than $M - 1$. (Why?)

Henceforth, when we talk about identification we mean exact identification or overidentification. There is no point in considering unidentified, or underidentified, equations because no matter how extensive the data, the structural parameters cannot be estimated. Besides, most simultaneous-equation systems in economics and finance are overidentified rather than underidentified, so we need not worry too much about underidentification. However, as shown in Chapter 20, parameters of overidentified as well as just identified equations can be estimated.

Which condition should one use in practice: Order or rank? For large simultaneous-equation models, applying the rank condition is a formidable task. Therefore, as Harvey notes,

> Fortunately, the order condition is usually sufficient to ensure identifiability, and although it is important to be aware of the rank condition, a failure to verify it will rarely result in disaster.[8]

*19.4 A Test of Simultaneity[9]

If there is no simultaneous equation, or **simultaneity problem,** the OLS estimators produce consistent and efficient estimators. On the other hand, if there is simultaneity, OLS estimators are not even consistent. In the presence of simultaneity, as we will show in Chapter 20, the methods of **two-stage least squares (2SLS)** and **instrumental variables (IV)** will give estimators that are consistent and efficient. Oddly, if we apply these alternative methods when there is in fact no simultaneity, these methods yield estimators that are consistent but not efficient (i.e., with smaller variance). This discussion suggests that we should check for the simultaneity problem before we discard OLS in favor of the alternatives.

As we showed earlier, the simultaneity problem arises because some of the regressors are endogenous and are therefore likely to be correlated with the disturbance, or error, term. Therefore, *a test of simultaneity is essentially a test of whether (an endogenous) regressor is correlated with the error term.* If it is, the simultaneity problem exists, in which case alternatives to OLS must be found; if it is not, we can use OLS. To find out which is the case in a concrete situation, we can use Hausman's specification error test.

Hausman Specification Test

A version of the Hausman specification error test that can be used for testing the simultaneity problem can be explained as follows:[10]

To fix ideas, consider the following two-equation model:

$$\textit{Demand function:} \quad Q_t^d = \alpha_0 + \alpha_1 P_t + \alpha_2 I_t + \alpha_3 R_t + u_{1t} \tag{19.4.1}$$

$$\textit{Supply function:} \quad Q_t^s = \beta_0 + \beta_1 P_t + u_{2t} \tag{19.4.2}$$

where P = price
Q = quantity
I = income
R = wealth
u's = error terms

Assume that I and R are exogenous. Of course, P and Q are endogenous.

*Optional.

[8]Andrew Harvey, *The Econometric Analysis of Time Series,* 2d ed., The MIT Press, Cambridge, Mass., 1990, p. 328.

[9]The following discussion draws from Robert S. Pindyck and Daniel L. Rubinfeld, *Econometric Models and Economic Forecasts,* 3d ed., McGraw-Hill, New York, 1991, pp. 303–305.

[10]J. A. Hausman, "Specification Tests in Econometrics," *Econometrica,* vol. 46, November 1976, pp. 1251–1271. See also A. Nakamura and M. Nakamura, "On the Relationship among Several Specification Error Tests Presented by Durbin, Wu, and Hausman," *Econometrica,* vol. 49, November 1981, pp. 1583–1588.

Now consider the supply function (19.4.2). If there is no simultaneity problem (i.e., P and Q are mutually independent), P_t and u_{2t} should be uncorrelated (why?). On the other hand, if there is simultaneity, P_t and u_{2t} will be correlated. To find out which is the case, the Hausman test proceeds as follows:

First, from Eqs. (19.4.1) and (19.4.2) we obtain the following reduced-form equations:

$$P_t = \Pi_0 + \Pi_1 I_t + \Pi_2 R_t + v_t \tag{19.4.3}$$

$$Q_t = \Pi_3 + \Pi_4 I_t + \Pi_5 R_t + w_t \tag{19.4.4}$$

where v and w are the reduced-form error terms. Estimating Eq. (19.4.3) by OLS we obtain

$$\hat{P}_t = \hat{\Pi}_0 + \hat{\Pi}_1 I_t + \hat{\Pi}_2 R_t \tag{19.4.5}$$

Therefore,

$$P_t = \hat{P}_t + \hat{v}_t \tag{19.4.6}$$

where $\hat{P}_t$ are estimated P_t and $\hat{v}_t$ are the estimated residuals. Now consider the following equation:

$$Q_t = \beta_0 + \beta_1 \hat{P}_t + \beta_1 \hat{v}_t + u_{2t} \tag{19.4.7}$$

Note: The coefficients of P_t and v_t are the same. The difference between this equation and the original supply equation is that it includes the additional variable $\hat{v}_t$, the residual from regression (19.4.3).

Now, if the null hypothesis is that there is no simultaneity, that is, P_t is not an endogenous variable, the correlation between $\hat{v}_t$ and u_{2t} should be zero, asymptotically. Thus, if we run the regression (19.4.7) and find that the coefficient of v_t in Eq. (19.4.7) is statistically zero, we can conclude that there is no simultaneity problem. Of course, this conclusion will be reversed if we find this coefficient to be statistically significant. In passing, note that Hausman's simultaneity test is also known as the *Hausman test of endogeneity:* In the present example we want to find out if P_t is endogenous. If it is, we have the simultaneity problem.

Essentially, then, the Hausman test involves the following steps:

Step 1. Regress P_t on I_t and R_t to obtain $\hat{v}_t$.

Step 2. Regress Q_t on $\hat{P}_t$ and $\hat{v}_t$ and perform a t test on the coefficient of $\hat{v}_t$. If it is significant, do not reject the hypothesis of simultaneity; otherwise, reject it.[11] For efficient estimation, however, Pindyck and Rubinfeld suggest regressing Q_t on P_t and $\hat{v}_t$.[12]

There are alternative ways to apply the Hausman test, which are given by way of an exercise.

EXAMPLE 19.5

Pindyck–Rubinfeld Model of Public Spending[13]

To study the behavior of U.S. state and local government expenditure, the authors developed the following simultaneous-equation model:

$$\text{EXP} = \beta_1 + \beta_2\text{AID} + \beta_3\text{INC} + \beta_4\text{POP} + u_i \tag{19.4.8}$$

$$\text{AID} = \delta_1 + \delta_2\text{EXP} + \delta_3\text{PS} + v_i \tag{19.4.9}$$

where
EXP = state and local government public expenditures
AID = level of federal grants-in-aid
INC = income of states
POP = state population
PS = population of primary and secondary school children
u and v = error terms

In this model, INC, POP, and PS are regarded as exogenous.

[11] If more than one endogenous regressor is involved, we will have to use the F test.

[12] Pindyck and Rubinfeld, op. cit., p. 304. *Note:* The regressor is P_t and *not* $\hat{P}_t$.

[13] Pindyck and Rubinfeld, op. cit., pp. 176–177. Notations slightly altered.

EXAMPLE 19.5 (*Continued*)

Because of the possibility of simultaneity between EXP and AID, the authors first regress AID on INC, POP, and PS (i.e., the reduced-form regression). Let the error term in this regression be w_i. From this regression the calculated residual is $\hat{w}_i$. The authors then regress EXP on AID, INC, POP, and $\hat{w}_i$, to obtain the following results:

$$\widehat{\text{EXP}} = -89.41 + 4.50\text{AID} + 0.00013\text{INC} - 0.518\text{POP} - 1.39\hat{w}_i$$
$$t = (-1.04) \quad (5.89) \quad (3.06) \quad (-4.63) \quad (-1.73) \qquad \textbf{(19.4.10)}^{14}$$
$$R^2 = 0.99$$

At the 5 percent level of significance, the coefficient of $\hat{w}_i$ is not statistically significant, and therefore, at this level, there is no simultaneity problem. However, at the 10 percent level of significance, it is statistically significant, raising the possibility that the simultaneity problem is present.

Incidentally, the OLS estimation of Eq. (19.4.8) is as follows:

$$\widehat{\text{EXP}} = -46.81 + 3.24\text{AID} + 0.00019\text{INC} - 0.597\text{POP}$$
$$t = (-0.56) \quad (13.64) \quad (8.12) \quad (-5.71) \qquad \textbf{(19.4.11)}$$
$$R^2 = 0.993$$

Notice an interesting feature of the results given in Eqs. (19.4.10) and (19.4.11): When simultaneity is explicitly taken into account, the AID variable is less significant although numerically it is greater in magnitude.

*19.5 Tests for Exogeneity

We noted earlier that it is the researcher's responsibility to specify which variables are endogenous and which are exogenous. This will depend on the problem at hand and the a priori information the researcher has. But is it possible to develop a statistical test of exogeneity, in the manner of Granger's causality test?

The Hausman test discussed in Section 19.4 can be utilized to answer this question. Suppose we have a three-equation model in three endogenous variables, Y_1, Y_2, and Y_3, and suppose there are three exogenous variables, X_1, X_2, and X_3. Further, suppose that the first equation of the model is

$$Y_{1i} = \beta_0 + \beta_2 Y_{2i} + \beta_3 Y_{3i} + \alpha_1 X_{1i} + u_{1i} \qquad \textbf{(19.5.1)}$$

If Y_2 and Y_3 are truly endogenous, we cannot estimate Eq. (19.5.1) by OLS (why?). But how do we find that out? We can proceed as follows. We obtain the reduced-form equations for Y_2 and Y_3 (*Note:* the reduced-form equations will have only predetermined variables on the right-hand side). From these reduced-form equations, we obtain $\hat{Y}_{2i}$ and $\hat{Y}_{3i}$, the predicted values of Y_{2i} and Y_{3i}, respectively. Then in the spirit of the Hausman test discussed earlier, we can estimate the following equation by OLS:

$$Y_{1i} = \beta_0 + \beta_2 Y_{2i} + \beta_3 Y_{3i} + \alpha_1 X_{1i} + \lambda_2 \hat{Y}_{2i} + \lambda_3 \hat{Y}_{3i} + u_{1i} \qquad \textbf{(19.5.2)}$$

Using the F test, we test the hypothesis that $\lambda_2 = \lambda_3 = 0$. If this hypothesis is rejected, Y_2 and Y_3 can be deemed endogenous, but if it is not rejected, they can be treated as exogenous. For a concrete example, see Exercise 19.16.

*Optional.

[14]As in footnote 12, the authors use AID rather than $\widehat{\text{AID}}$ as the regressor.

Summary and Conclusions

1. The problem of identification precedes the problem of estimation.
2. The identification problem asks whether one can obtain unique numerical estimates of the structural coefficients from the estimated reduced-form coefficients.
3. If this can be done, an equation in a system of simultaneous equations is identified. If this cannot be done, that equation is un- or under-identified.
4. An identified equation can be just identified or overidentified. In the former case, unique values of structural coefficients can be obtained; in the latter, there may be more than one value for one or more structural parameters.
5. The identification problem arises because the same set of data may be compatible with different sets of structural coefficients, that is, different models. Thus, in the regression of price on quantity only, it is difficult to tell whether one is estimating the supply function or the demand function, because price and quantity enter both equations.
6. To assess the identifiability of a structural equation, one may apply the technique of **reduced-form equations,** which expresses an endogenous variable solely as a function of predetermined variables.
7. However, this time-consuming procedure can be avoided by resorting to either the **order condition** or the **rank condition** of identification. Although the order condition is easy to apply, it provides only a necessary condition for identification. On the other hand, the rank condition is both a necessary and sufficient condition for identification. If the rank condition is satisfied, the order condition is satisfied, too, although the converse is not true. In practice, though, the order condition is generally adequate to ensure identifiability.
8. In the presence of simultaneity, OLS is generally not applicable, as was shown in Chapter 18. But if one wants to use it nonetheless, it is imperative to test for simultaneity explicitly. The **Hausman specification test** can be used for this purpose.
9. Although in practice deciding whether a variable is endogenous or exogenous is a matter of judgment, one can use the Hausman specification test to determine whether a variable or group of variables is endogenous or exogenous.
10. Although they are in the same family, the concepts of causality and exogeneity are different and one may not necessarily imply the other. In practice it is better to keep those concepts separate (see Section 17.14).

EXERCISES

Questions

19.1. Show that the two definitions of the order condition of identification (see Section 19.3) are equivalent.

19.2. Deduce the structural coefficients from the reduced-form coefficients given in Eqs. (19.2.25) and (19.2.27).

19.3. Obtain the reduced form of the following models and determine in each case whether the structural equations are unidentified, just identified, or overidentified:

a. Chap. 18, Example 18.2.

b. Chap. 18, Example 18.3.

c. Chap. 18, Example 18.6.

19.4. Check the identifiability of the models of Exercise 19.3 by applying both the order and rank conditions of identification.

19.5. In the model (19.2.22) of the text it was shown that the supply equation was overidentified. What restrictions, if any, on the structural parameters will make this equation just identified? Justify the restrictions you impose.

19.6. From the model

$$Y_{1t} = \beta_{10} + \beta_{12}Y_{2t} + \gamma_{11}X_{1t} + u_{1t}$$
$$Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + \gamma_{22}X_{2t} + u_{2t}$$

the following reduced-form equations are obtained:

$$Y_{1t} = \Pi_{10} + \Pi_{11}X_{1t} + \Pi_{12}X_{2t} + w_t$$
$$Y_{2t} = \Pi_{20} + \Pi_{21}X_{1t} + \Pi_{22}X_{2t} + v_t$$

a. Are the structural equations identified?

b. What happens to identification if it is known a priori that $\gamma_{11} = 0$?

19.7. Refer to Exercise 19.6. The estimated reduced-form equations are as follows:

$$Y_{1t} = 4 + 3X_{1t} + 8X_{2t}$$
$$Y_{2t} = 2 + 6X_{1t} + 10X_{2t}$$

a. Obtain the values of the structural parameters.

b. How would you test the null hypothesis that $\gamma_{11} = 0$?

19.8. The model

$$Y_{1t} = \beta_{10} + \beta_{12}Y_{2t} + \gamma_{11}X_{1t} + u_{1t}$$
$$Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + u_{2t}$$

produces the following reduced-form equations:

$$Y_{1t} = 4 + 8X_{1t}$$
$$Y_{2t} = 2 + 12X_{1t}$$

a. Which structural coefficients, if any, can be estimated from the reduced-form coefficients? Demonstrate your contention.

b. How does the answer to (*a*) change if it is known a priori that (1) $\beta_{12} = 0$ and (2) $\beta_{10} = 0$?

19.9. Determine whether the structural equations of the model given in Exercise 18.8 are identified.

19.10. Refer to Exercise 18.7 and find out which structural equations can be identified.

19.11. Table 19.3 is a model in five equations with five endogenous variables Y and four exogenous variables X:

TABLE 19.3

	Coefficients of the Variables								
Equation No.	Y_1	Y_2	Y_3	Y_4	Y_5	X_1	X_2	X_3	X_4
1	1	β_{12}	0	β_{14}	0	γ_{11}	0	0	γ_{14}
2	0	1	β_{23}	β_{24}	0	0	γ_{22}	γ_{23}	0
3	β_{31}	0	1	β_{34}	β_{35}	0	0	γ_{33}	γ_{34}
4	0	β_{42}	0	1	0	γ_{41}	0	γ_{43}	0
5	β_{51}	0	0	β_{54}	1	0	γ_{52}	γ_{53}	0

Determine the identifiability of each equation with the aid of the order and rank conditions of identifications.

19.12. Consider the following extended Keynesian model of income determination:

Consumption function:	$C_t = \beta_1 + \beta_2 Y_t - \beta_3 T_t + u_{1t}$
Investment function:	$I_t = \alpha_0 + \alpha_1 Y_{t-1} + u_{2t}$
Taxation function:	$T_t = \gamma_0 + \gamma_1 Y_t + u_{3t}$
Income identity:	$Y_t = C_t + I_t + G_t$

where C = consumption expenditure
Y = income
I = investment
T = taxes
G = government expenditure
u's = the disturbance terms

In the model the endogenous variables are C, I, T, and Y and the predetermined variables are G and Y_{t-1}.

By applying the order condition, check the identifiability of each of the equations in the system and of the system as a whole. What would happen if r_t, the interest rate, assumed to be exogenous, were to appear on the right-hand side of the investment function?

19.13. Refer to the data given in Table 18.1 of Chapter 18. Using these data, estimate the reduced-form regressions (19.1.2) and (19.1.4). Can you estimate β_0 and β_1? Show your calculations. Is the model identified? Why or why not?

19.14. Suppose we propose yet another definition of the order condition of identifiability:

$$K \geq m + k - 1$$

which states that the number of predetermined variables in the system can be no less than the number of unknown coefficients in the equation to be identified. Show that this definition is equivalent to the two other definitions of the order condition given in the text.

19.15. A simplified version of Suits's model of the watermelon market is as follows:*

Demand equation: $P_t = \alpha_0 + \alpha_1(Q_t/N_t) + \alpha_2(Y_t/N_t) + \alpha_3 F_t + u_{1t}$

Crop supply function: $Q_t = \beta_0 + \beta_1(P_t/W_t) + \beta_2 P_{t-1} + \beta_3 C_{t-1} + \beta_4 T_{t-1} + u_{2t}$

where P = price
(Q/N) = per capita quantity demanded
(Y/N) = per capita income
F_t = freight costs
(P/W) = price relative to the farm wage rate
C = price of cotton
T = price of other vegetables
N = population

P and Q are the endogenous variables.

a. Obtain the reduced form.

b. Determine whether the demand, the supply, or both functions are identified.

Empirical Exercises

19.16. Consider the following demand-and-supply model for money:

Money demand: $M_t^d = \beta_0 + \beta_1 Y_t + \beta_2 R_t + \beta_3 P_t + u_{1t}$

Money supply: $M_t^s = \alpha_0 + \alpha_1 Y_t + u_{2t}$

*D. B. Suits, "An Econometric Model of the Watermelon Market," *Journal of Farm Economics,* vol. 37, 1955, pp. 237–251.

TABLE 19.4
Money, GDP, Interest Rate, and Consumer Price Index, United States, 1970–2006

Source: *Economic Report of the President,* 2007, Tables B-2, B-60, B-69, B-73.

Observation	M_2	GDP	TBRATE	CPI
1970	626.5	3,771.9	6.458	38.8
1971	710.3	3,898.6	4.348	40.5
1972	802.3	4,105.0	4.071	41.8
1973	855.5	4,341.5	7.041	44.4
1974	902.1	4,319.6	7.886	49.3
1975	1,016.2	4,311.2	5.838	53.8
1976	1,152.0	4,540.9	4.989	56.9
1977	1,270.3	4,750.5	5.265	60.6
1978	1,366.0	5,015.0	7.221	65.2
1979	1,473.7	5,173.4	10.041	72.6
1980	1,599.8	5,161.7	11.506	82.4
1981	1,755.5	5,291.7	14.029	90.9
1982	1,910.1	5,189.3	10.686	96.5
1983	2,126.4	5,423.8	8.63	99.6
1984	2,309.8	5,813.6	9.58	103.9
1985	2,495.5	6,053.7	7.48	107.6
1986	2,732.2	6,263.6	5.98	109.6
1987	2,831.3	6,475.1	5.82	113.6
1988	2,994.3	6,742.7	6.69	118.3
1989	3,158.3	6,981.4	8.12	124.0
1990	3,277.7	7,112.5	7.51	130.7
1991	3,378.3	7,100.5	5.42	136.2
1992	3,431.8	7,336.6	3.45	140.3
1993	3,482.5	7,532.7	3.02	144.5
1994	3,498.5	7,835.5	4.29	148.2
1995	3,641.7	8,031.7	5.51	152.4
1996	3,820.5	8,328.9	5.02	156.9
1997	4,035.0	8,703.5	5.07	160.5
1998	4,381.8	9,066.9	4.81	163.0
1999	4,639.2	9,470.3	4.66	166.6
2000	4,921.7	9,817.0	5.85	172.2
2001	5,433.5	9,890.7	3.45	177.1
2002	5,779.2	10,048.8	1.62	179.9
2003	6,071.2	10,301.0	1.02	184.0
2004	6,421.6	10,675.8	1.38	188.9
2005	6,691.7	11,003.4	3.16	195.3
2006	7,035.5	11,319.4	4.73	201.6

Notes: $M_2 = M_2$ Money supply (billions of dollars).
GDP = gross domestic product (billions of dollars).
TBRATE = 3-month Treasury bill rate, %.
CPI = Consumer Price Index (1982–1984 = 100).

where M = money
Y = income
R = rate of interest
P = price
u's = error terms

Assume that R and P are exogenous and M and Y are endogenous. Table 19.4 gives data on M (M_2 definition), Y (GDP), R (3-month Treasury bill rate) and P (Consumer Price Index), for the United States for 1970–2006.

a. Is the demand function identified?

b. Is the supply function identified?

c. Obtain the expressions for the reduced-form equations for M and Y.

d. Apply the test of simultaneity to the supply function.

e. How would we find out if Y in the money supply function is in fact endogenous?

19.17. The Hausman test discussed in the text can also be conducted in the following way. Consider Eq. (19.4.7):

$$Q_t = \beta_0 + \beta_1 P_t + \beta_1 v_t + u_{2t}$$

a. Since P_t and v_t have the same coefficients, how would you test that in a given application that is indeed the case? What are the implications of this?

b. Since P_t is uncorrelated with u_{2t} by design (why?), one way to find out if P_t is exogenous is to see if v_t is correlated with u_{2t}. How would you go about testing this? Which test do you use? (*Hint:* Substitute P_t from [19.4.6] into Eq. [19.4.7].)

Chapter 20

Simultaneous-Equation Methods

Having discussed the nature of the simultaneous-equation models in the previous two chapters, in this chapter we turn to the problem of estimation of the parameters of such models. At the outset it may be noted that the estimation problem is rather complex because there are a variety of estimation techniques with varying statistical properties. In view of the introductory nature of this text, we shall consider only a few of these techniques. Our discussion will be simple and often heuristic, the finer points being left to the references.

20.1 Approaches to Estimation

If we consider the general M equations model in M endogenous variables given in Eq. (19.1.1), we may adopt two approaches to estimate the structural equations, namely, single-equation methods, also known as **limited information methods,** and system methods, also known as **full information methods.** In the single-equation methods to be considered shortly, we estimate each equation in the system (of simultaneous equations) individually, taking into account any restrictions placed on that equation (such as exclusion of some variables) without worrying about the restrictions on the other equations in the system,[1] hence the name *limited information methods.* In the system methods, on the other hand, we estimate all the equations in the model simultaneously, taking due account of all restrictions on such equations by the omission or absence of some variables (recall that for identification such restrictions are essential), hence the name *full information methods*.

As an example, consider the following four-equations model:

$$
\begin{aligned}
Y_{1t} &= \beta_{10} + \qquad\quad + \beta_{12}Y_{2t} + \beta_{13}Y_{3t} + \qquad + \gamma_{11}X_{1t} + \qquad\qquad\qquad + u_{1t} \\
Y_{2t} &= \beta_{20} + \qquad\qquad\qquad\quad + \beta_{23}Y_{3t} \qquad + \gamma_{21}X_{1t} + \gamma_{22}X_{2t} \qquad + u_{2t} \\
Y_{3t} &= \beta_{30} + \beta_{31}Y_{1t} + \qquad\qquad\qquad + \beta_{34}Y_{4t} + \gamma_{31}X_{1t} + \gamma_{32}X_{2t} + \qquad + u_{3t} \\
Y_{4t} &= \beta_{40} + \qquad\quad + \beta_{42}Y_{2t} \qquad\qquad\qquad\qquad\qquad\qquad + \gamma_{43}X_{3t} + u_{4t}
\end{aligned}
\tag{20.1.1}
$$

[1]For the purpose of identification, however, information provided by other equations will have to be taken into account. But as noted in Chapter 19, estimation is possible only in the case of (fully or over-) identified equations. In this chapter we assume that the identification problem is solved using the techniques of Chapter 19.

where the Y's are the endogenous variables and the X's are the exogenous variables. If we are interested in estimating, say, the third equation, the single-equation methods will consider this equation only, noting that variables Y_2 and X_3 are excluded from it. In the systems methods, on the other hand, we try to estimate all four equations simultaneously, taking into account all the restrictions imposed on the various equations of the system.

To preserve the spirit of simultaneous-equation models, ideally one should use the systems method, such as the **full information maximum likelihood (FIML) method.**[2] In practice, however, such methods are not commonly used for a variety of reasons. First, the computational burden is enormous. For example, the comparatively small (20 equations) 1955 Klein–Goldberger model of the U.S. economy had 151 nonzero coefficients, of which the authors estimated only 51 coefficients using the time series data. The Brookings-Social Science Research Council (SSRC) econometric model of the U.S. economy published in 1965 initially had 150 equations.[3] Although such elaborate models may furnish finer details of the various sectors of the economy, the computations are a stupendous task even in these days of high-speed computers, not to mention the cost involved. Second, the systems methods, such as FIML, lead to solutions that are highly nonlinear in the parameters and are therefore often difficult to determine. Third, if there is a specification error (say, a wrong functional form or exclusion of relevant variables) in one or more equations of the system, that error is transmitted to the rest of the system. As a result, the systems methods become very sensitive to specification errors.

In practice, therefore, single-equation methods are often used. As Klein puts it,

> Single equation methods, in the context of a simultaneous system, may be less sensitive to specification error in the sense that those parts of the system that are correctly specified may not be affected appreciably by errors in specification in another part.[4]

In the rest of the chapter we shall deal with single-equation methods only. Specifically, we shall discuss the following single-equation methods:

1. Ordinary least squares (OLS)
2. Indirect least squares (ILS)
3. Two-stage least squares (2SLS)

20.2 Recursive Models and Ordinary Least Squares

We saw in Chapter 18 that, because of the interdependence between the stochastic disturbance term and the endogenous explanatory variable(s), the OLS method is inappropriate for the estimation of an equation in a system of simultaneous equations. If applied erroneously, then, as we saw in Section 18.3, the estimators are not only biased (in small samples) but also inconsistent; that is, the bias does not disappear no matter how large the sample size. There is, however, one situation where OLS can be applied appropriately even in the context of simultaneous equations. This is the case of the **recursive, triangular,** or

[2] For a simple discussion of this method, see Carl F. Christ, *Econometric Models and Methods*, John Wiley & Sons, New York, 1966, pp. 395–401.

[3] James S. Duesenberry, Gary Fromm, Lawrence R. Klein, and Edwin Kuh, eds., *A Quarterly Model of the United States Economy*, Rand McNally, Chicago, 1965.

[4] Lawrence R. Klein, *A Textbook of Econometrics,* 2d ed., Prentice Hall, Englewood Cliffs, NJ, 1974, p. 150.

causal models. To see the nature of these models, consider the following three-equation system:

$$
\begin{aligned}
Y_{1t} &= \beta_{10} & &+ \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + u_{1t} \\
Y_{2t} &= \beta_{20} + \beta_{21}Y_{1t} & &+ \gamma_{21}X_{1t} + \gamma_{22}X_{2t} + u_{2t} \qquad \textbf{(20.2.1)} \\
Y_{3t} &= \beta_{30} + \beta_{31}Y_{1t} + \beta_{32}Y_{2t} & &+ \gamma_{31}X_{1t} + \gamma_{32}X_{2t} + u_{3t}
\end{aligned}
$$

where, as usual, the Y's and the X's are, respectively, the endogenous and exogenous variables. The disturbances are such that

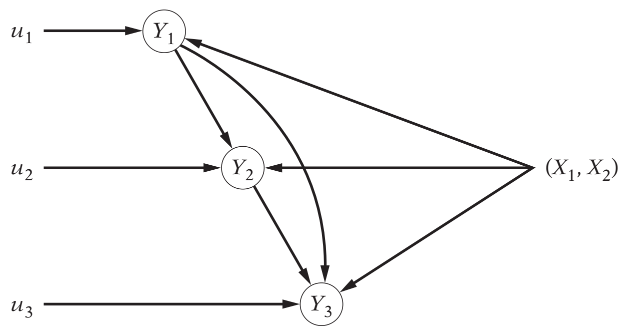
$$\text{cov}\,(u_{1t}, u_{2t}) = \text{cov}\,(u_{1t}, u_{3t}) = \text{cov}\,(u_{2t}, u_{3t}) = 0$$

that is, the same-period disturbances in different equations are uncorrelated (technically, this is the assumption of **zero contemporaneous correlation**).

Now consider the first equation of (20.2.1). Since it contains only the exogenous variables on the right-hand side and since by assumption they are uncorrelated with the disturbance term u_{1t}, this equation satisfies the critical assumption of the classical OLS, namely, uncorrelatedness between the explanatory variables and the stochastic disturbances. Hence, OLS can be applied straightforwardly to this equation. Next consider the second equation of (20.2.1), which contains the endogenous variable Y_1 as an explanatory variable along with the nonstochastic X's. Now OLS can also be applied to this equation, provided Y_{1t} and u_{2t} are uncorrelated. Is this so? The answer is yes because u_1, which affects Y_1, is by assumption uncorrelated with u_2. Therefore, for all practical purposes, Y_1 is a predetermined variable insofar as Y_2 is concerned. Hence, one can proceed with OLS estimation of this equation. Carrying this argument a step further, we can also apply OLS to the third equation in (20.2.1) because both Y_1 and Y_2 are uncorrelated with u_3.

Thus, in the recursive system OLS can be applied to each equation separately. Actually, we do not have a simultaneous-equation problem in this situation. From the structure of such systems, it is clear that there is no interdependence among the endogenous variables. Thus, Y_1 affects Y_2, but Y_2 does not affect Y_1. Similarly, Y_1 and Y_2 influence Y_3 without, in turn, being influenced by Y_3. In other words, each equation exhibits a unilateral causal dependence, hence the name causal models.[5] Schematically, we have Figure 20.1.

FIGURE 20.1
Recursive model.



[5]The alternative name *triangular* stems from the fact that if we form the matrix of the coefficients of the endogenous variables given in Eq. (20.2.1), we obtain the following triangular matrix:

	Y_1	Y_2	Y_3
Equation 1	1	0	0
Equation 2	β_{21}	1	0
Equation 3	β_{31}	β_{32}	1

Note that the entries above the main diagonal are zeros (why?).

As an example of a recursive system, one may postulate the following model of wage and price determination:

$$\text{Price equation:} \quad \dot{P}_t = \beta_{10} + \beta_{11}\dot{W}_{t-1} + \beta_{12}\dot{R}_t + \beta_{13}\dot{M}_t + \beta_{14}\dot{L}_t + u_{1t}$$
$$\text{Wage equation:} \quad \dot{W}_t = \beta_{20} + \beta_{21}\text{UN}_t + \beta_{32}\dot{P}_t + u_{2t} \qquad \textbf{(20.2.2)}$$

where $\dot{P}$ = rate of change of price per unit of output
$\dot{W}$ = rate of change of wages per employee
$\dot{R}$ = rate of change of price of capital
$\dot{M}$ = rate of change of import prices
$\dot{L}$ = rate of change of labor productivity
UN = unemployment rate, %[6]

The price equation postulates that the rate of change of price in the current period is a function of the rates of change in the prices of capital and of raw material, the rate of change in labor productivity, and the rate of change in wages in the previous period. The wage equation shows that the rate of change in wages in the current period is determined by the current period rate of change in price and the unemployment rate. It is clear that the causal chain runs from $\dot{W}_{t-1} \rightarrow \dot{P}_t \rightarrow \dot{W}_t$, and hence OLS may be applied to estimate the parameters of the two equations individually.

Although recursive models have proved to be useful, most simultaneous-equation models do not exhibit such a unilateral cause-and-effect relationship. Therefore, OLS, in general, is inappropriate to estimate a single equation in the context of a simultaneous-equation model.[7]

There are some who argue that, although OLS is generally inapplicable to simultaneous-equation models, one can use it, if only as a standard or norm of comparison. That is, one can estimate a structural equation by OLS, with the resulting properties of biasedness, inconsistency, etc. Then the same equation may be estimated by other methods especially designed to handle the simultaneity problem and the results of the two methods compared, at least qualitatively. In many applications the results of the inappropriately applied OLS may not differ very much from those obtained by more sophisticated methods, as we shall see later. In principle, one should not have much objection to the production of the results based on OLS so long as estimates based on alternative methods devised for simultaneous-equation models are also given. In fact, this approach might give us some idea about how badly OLS does in situations when it is applied inappropriately.[8]

[6]*Note:* The dotted symbol means "time derivative." For example, $\dot{P} = dP/dt$. For discrete time series, dP/dt is sometimes approximated by $\Delta P/\Delta t$, where the symbol Δ is the first difference operator, which was originally introduced in Chapter 12.

[7]It is important to keep in mind that we are assuming that the disturbances across equations are contemporaneously uncorrelated. If this is not the case, we may have to resort to the Zellner SURE (seemingly unrelated regressions) estimation technique to estimate the parameters of the recursive system. See A. Zellner, "An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias," *Journal of the American Statistical Association,* vol. 57, 1962, pp. 348–368.

[8]It may also be noted that in small samples the alternative estimators, like the OLS estimators, are also biased. But the OLS estimator has the "virtue" that it has minimum variance among these alternative estimators. But this is true of small samples only.

20.3 Estimation of a Just Identified Equation: The Method of Indirect Least Squares (ILS)

For a just or exactly identified structural equation, the method of obtaining the estimates of the structural coefficients from the OLS estimates of the reduced-form coefficients is known as the **method of indirect least squares (ILS),** and the estimates thus obtained are known as the **indirect least-squares estimates.** ILS involves the following three steps:

Step 1. We first obtain the reduced-form equations. As noted in Chapter 19, these reduced-form equations are obtained from the structural equations in such a manner that the dependent variable in each equation is the only endogenous variable and is a function solely of the predetermined (exogenous or lagged endogenous) variables and the stochastic error term(s).

Step 2. We apply OLS to the reduced-form equations individually. This operation is permissible since the explanatory variables in these equations are predetermined and hence uncorrelated with the stochastic disturbances. The estimates thus obtained are consistent.[9]

Step 3. We obtain estimates of the original structural coefficients from the estimated reduced-form coefficients obtained in Step 2. As noted in Chapter 19, if an equation is exactly identified, there is a one-to-one correspondence between the structural and reduced-form coefficients; that is, one can derive unique estimates of the former from the latter.

As this three-step procedure indicates, the name ILS derives from the fact that structural coefficients (the object of primary enquiry in most cases) are obtained indirectly from the OLS estimates of the reduced-form coefficients.

An Illustrative Example

Consider the demand-and-supply model introduced in Section 19.2, which for convenience is given below with a slight change in notation:

$$\text{Demand function:} \quad Q_t = \alpha_0 + \alpha_1 P_t + \alpha_2 X_t + u_{1t} \tag{20.3.1}$$

$$\text{Supply function:} \quad Q_t = \beta_0 + \beta_1 P_t + u_{2t} \tag{20.3.2}$$

where Q = quantity
P = price
X = income or expenditure

Assume that X is exogenous. As noted previously, the supply function is exactly identified whereas the demand function is not identified.

The reduced-form equations corresponding to the preceding structural equations are

$$P_t = \Pi_0 + \Pi_1 X_t + w_t \tag{20.3.3}$$

$$Q_t = \Pi_2 + \Pi_3 X_t + v_t \tag{20.3.4}$$

[9]In addition to being consistent, the estimates "may be best unbiased and/or asymptotically efficient, depending respectively upon whether (*i*) the z's [= X's] are exogenous and not merely predetermined [i.e., do not contain lagged values of endogenous variables] and/or (*ii*) the distribution of the disturbances is normal." See W. C. Hood and Tjalling C. Koopmans, *Studies in Econometric Method,* John Wiley & Sons, New York, 1953, p. 133.

where the Π's are the reduced-form coefficients and are (nonlinear) combinations of the structural coefficients, as shown in Eqs. (19.2.16) and (19.2.18), and where w and v are linear combinations of the structural disturbances u_1 and u_2.

Notice that each reduced-form equation contains only one endogenous variable, which is the dependent variable and which is a function solely of the exogenous variable X (income) and the stochastic disturbances. Hence, the parameters of the preceding reduced-form equations may be estimated by OLS. These estimates are

$$\hat{\Pi}_1 = \frac{\sum p_t x_t}{\sum x_t^2} \tag{20.3.5}$$

$$\hat{\Pi}_0 = \bar{P} - \hat{\Pi}_1 \bar{X} \tag{20.3.6}$$

$$\hat{\Pi}_3 = \frac{\sum q_t x_t}{\sum x_t^2} \tag{20.3.7}$$

$$\hat{\Pi}_2 = \bar{Q} - \hat{\Pi}_3 \bar{X} \tag{20.3.8}$$

where the lowercase letters, as usual, denote deviations from sample means and where $\bar{Q}$ and $\bar{P}$ are the sample mean values of Q and P. As noted previously, the $\hat{\Pi}_i$'s are consistent estimators and under appropriate assumptions are also minimum variance unbiased or asymptotically efficient (see footnote 9).

Since our primary objective is to determine the structural coefficients, let us see if we can estimate them from the reduced-form coefficients. Now as shown in Section 19.2, the supply function is exactly identified. Therefore, its parameters can be estimated uniquely from the reduced-form coefficients as follows:

$$\beta_0 = \Pi_2 - \beta_1 \Pi_0 \qquad \text{and} \qquad \beta_1 = \frac{\Pi_3}{\Pi_1}$$

Hence, the estimates of these parameters can be obtained from the estimates of the reduced-form coefficients as

$$\hat{\beta}_0 = \hat{\Pi}_2 - \hat{\beta}_1 \hat{\Pi}_0 \tag{20.3.9}$$

$$\hat{\beta}_1 = \frac{\hat{\Pi}_3}{\hat{\Pi}_1} \tag{20.3.10}$$

which are the ILS estimators. Note that the parameters of the demand function cannot be thus estimated (however, see Exercise 20.13).

To give some numerical results, we obtained the data shown in Table 20.1. First we estimate the reduced-form equations, regressing separately price and quantity on per capita real consumption expenditure. The results are as follows:

$$\begin{aligned} \hat{P}_t &= 90.9601 + 0.0007X_t \\ \text{se} &= (4.0517) \quad (0.0002) \\ t &= (22.4499) \quad (3.0060) \qquad R^2 = (0.2440) \end{aligned} \tag{20.3.11}$$

$$\begin{aligned} \hat{Q}_t &= 59.7618 + 0.0020X_t \\ \text{se} &= (1.5600) \quad (0.00009) \\ t &= (38.3080) \quad (20.9273) \qquad R^2 = 0.9399 \end{aligned} \tag{20.3.12}$$

Using Eqs. (20.3.9) and (20.3.10), we obtain these ILS estimates:

$$\hat{\beta}_0 = -183.7043 \tag{20.3.13}$$

$$\hat{\beta}_1 = 2.6766 \tag{20.3.14}$$

TABLE 20.1
Crop Production, Crop Prices, and per Capita Personal Consumption Expenditures, 2007 Dollars, United States, 1975–2004

Source: *Economic Report of the President,* 2007. Data on Q (Table B-99), on P (Table B-101), and on X (Table B-31).

Observation	Index of Crop Production (1996 = 100), Q	Index of Crop Prices Received by Farmers (1990–1992 = 100), P	Real per Capita Personal Consumption Expenditure, X
1975	66	88	4,789
1976	67	87	5,282
1977	71	83	5,804
1978	73	89	6,417
1979	78	98	7,073
1980	75	107	7,716
1981	81	111	8,439
1982	82	98	8,945
1983	71	108	9,775
1984	81	111	10,589
1985	85	98	11,406
1986	82	87	12,048
1987	84	86	12,766
1988	80	104	13,685
1989	86	109	14,546
1990	90	103	15,349
1991	90	101	15,722
1992	96	101	16,485
1993	91	102	17,204
1994	101	105	18,004
1995	96	112	18,665
1996	100	127	19,490
1997	104	115	20,323
1998	105	107	21,291
1999	108	97	22,491
2000	108	96	23,862
2001	108	99	24,722
2002	107	105	25,501
2003	108	111	26,463
2004	112	117	27,937

Therefore, the estimated ILS regression is[10]

$$\hat{Q}_t = -183.7043 + 2.6766P_t \qquad (20.3.15)$$

For comparison, we give the results of the (inappropriately applied) OLS regression of Q on P:

$$\begin{aligned} \hat{Q}_t &= \ 20.89 + \ 0.673P_t \\ \text{se} &= (23.04) \quad (0.2246) \\ t &= \ (0.91) \quad (2.99) \qquad R^2 = 0.2430 \end{aligned} \qquad (20.3.16)$$

These results show how OLS can distort the "true" picture when it is applied in inappropriate situations.

[10]We have not presented the standard errors of the estimated structural coefficients because, as noted previously, these coefficients are generally nonlinear functions of the reduced-form coefficients and there is no simple method of estimating their standard errors from the standard errors of the reduced-form coefficients. For large-sample size, however, standard errors of the structural coefficients can be obtained approximately. For details, see Jan Kmenta, *Elements of Econometrics,* Macmillan, New York, 1971, p. 444.

Properties of ILS Estimators

We have seen that the estimators of the reduced-form coefficients are consistent and under appropriate assumptions also best unbiased or asymptotically efficient (see footnote 9). Do these properties carry over to the ILS estimators? It can be shown that the ILS estimators inherit all the asymptotic properties of the reduced-form estimators, such as consistency and asymptotic efficiency. But (the small sample) properties such as unbiasedness do not generally hold true. It is shown in Appendix 20A, Section 20A.1, that the ILS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ of the supply function given previously are biased but the bias disappears as the sample size increases indefinitely (that is, the estimators are consistent).[11]

20.4 Estimation of an Overidentified Equation: The Method of Two-Stage Least Squares (2SLS)

Consider the following model:

Income function:
$$Y_{1t} = \beta_{10} + \qquad + \beta_{11} Y_{2t} + \gamma_{11} X_{1t} + \gamma_{12} X_{2t} + u_{1t} \tag{20.4.1}$$

Money supply function:
$$Y_{2t} = \beta_{20} + \beta_{21} Y_{1t} \qquad\qquad + u_{2t} \tag{20.4.2}$$

where Y_1 = income
Y_2 = stock of money
X_1 = investment expenditure
X_2 = government expenditure on goods and services

The variables X_1 and X_2 are exogenous.

The income equation, a hybrid of quantity-theory–Keynesian approaches to income determination, states that income is determined by money supply, investment expenditure, and government expenditure. The *money supply function* postulates that the stock of money is determined (by the Federal Reserve System) on the basis of the level of income. Obviously, we have a simultaneous-equation problem, which can be checked by the simultaneity test discussed in Chapter 19.

Applying the order condition of identification, we can see that the income equation is underidentified whereas the money supply equation is overidentified. There is not much that can be done about the income equation short of changing the model specification. The overidentified money supply function may not be estimated by ILS because there are two estimates of β_{21} (the reader should verify this via the reduced-form coefficients).

As a matter of practice, one may apply OLS to the money supply equation, but the estimates thus obtained will be inconsistent in view of the likely correlation between the stochastic explanatory variable Y_1 and the stochastic disturbance term u_2. Suppose, however, we find a "proxy" for the stochastic explanatory variable Y_1 such that, although "resembling" Y_1 (in the sense that it is highly correlated with Y_1), it is uncorrelated with u_2. Such a proxy is also known as an **instrumental variable** (see Chapter 17). If one can find such a proxy, OLS can be used straightforwardly to estimate the money supply function.

[11]Intuitively this can be seen as follows: $E(\hat{\beta}_1) = \beta_1$ if $E(\hat{\Pi}_3/\hat{\Pi}_1) = (\Pi_3/\Pi_1)$. Now even if $E(\hat{\Pi}_3) = \Pi_3$ and $E(\hat{\Pi}_1), = \Pi_1$, it can be shown that $E(\hat{\Pi}_3/\hat{\Pi}_1) \neq E(\hat{\Pi}_3)/E(\hat{\Pi}_1)$; that is, the expectation of the ratio of two variables is not equal to the ratio of the expectations of the two variables. However, as shown in Appendix 20A.1, plim $(\hat{\Pi}_3/\hat{\Pi}_1)$ = plim $(\hat{\Pi}_3)$/plim $(\hat{\Pi}_1) = \Pi_3/\Pi_1$ since $\hat{\Pi}_3$ and $\hat{\Pi}_1$ are consistent estimators.

But how does one obtain such an instrumental variable? One answer is provided by the **two-stage least squares** (2SLS), developed independently by Henri Theil[12] and Robert Basmann.[13] As the name indicates, the method involves two successive applications of OLS. The process is as follows:

Stage 1. To get rid of the likely correlation between Y_1 and u_2, regress first Y_1 on *all* the predetermined variables in the *whole system,* not just that equation. In the present case, this means regressing Y_1 on X_1 and X_2 as follows:

$$Y_{1t} = \hat{\Pi}_0 + \hat{\Pi}_1 X_{1t} + \hat{\Pi}_2 X_{2t} + \hat{u}_t \quad \textbf{(20.4.3)}$$

where $\hat{u}_t$ are the usual OLS residuals. From Eq. (20.4.3) we obtain

$$\hat{Y}_{1t} = \hat{\Pi}_0 + \hat{\Pi}_1 X_{1t} + \hat{\Pi}_2 X_{2t} \quad \textbf{(20.4.4)}$$

where $\hat{Y}_{1t}$ is an estimate of the mean value of Y conditional upon the fixed X's. Note that Eq. (20.4.3) is nothing but a reduced-form regression because only the exogenous or predetermined variables appear on the right-hand side.

Equation (20.4.3) can now be expressed as

$$Y_{1t} = \hat{Y}_{1t} + \hat{u}_t \quad \textbf{(20.4.5)}$$

which shows that the stochastic Y_1 consists of two parts: $\hat{Y}_{1t}$, which is a linear combination of the nonstochastic X's, and a random component $\hat{u}_t$. Following the OLS theory, $\hat{Y}_{1t}$ and $\hat{u}_t$ are uncorrelated. (Why?)

Stage 2. The overidentified money supply equation can now be written as

$$\begin{aligned} Y_{2t} &= \beta_{20} + \beta_{21}(\hat{Y}_{1t} + \hat{u}_t) + u_{2t} \\ &= \beta_{20} + \beta_{21}\hat{Y}_{1t} + (u_{2t} + \beta_{21}\hat{u}_t) \\ &= \beta_{20} + \beta_{21}\hat{Y}_{1t} + u_t^* \end{aligned} \quad \textbf{(20.4.6)}$$

where $u_t^* = u_{2t} + \beta_{21}\hat{u}_t$.

Comparing Eq. (20.4.6) with Eq. (20.4.2), we see that they are very similar in appearance, the only difference being that Y_1 is replaced by $\hat{Y}_1$. What is the advantage of Eq. (20.4.6)? It can be shown that although Y_1 in the original money supply equation is correlated or likely to be correlated with the disturbance term u_2 (hence rendering OLS inappropriate), $\hat{Y}_{1t}$ in Eq. (20.4.6) is uncorrelated with u_t^* *asymptotically,* that is, in the large sample (or more accurately, as the sample size increases indefinitely). As a result, OLS can be applied to Eq. (20.4.6), which will give consistent estimates of the parameters of the money supply function.[14]

[12]Henri Theil, "Repeated Least-Squares Applied to Complete Equation Systems," The Hague: The Central Planning Bureau, The Netherlands, 1953 (mimeographed).

[13]Robert L. Basmann, "A Generalized Classical Method of Linear Estimation of Coefficients in a Structural Equation," *Econometrica,* vol. 25, 1957, pp. 77–83.

[14]But note that in small samples $\hat{Y}_{1t}$ is likely to be correlated with u_i^*. The reason is as follows: From Eq. (20.4.4) we see that $\hat{Y}_{1t}$ is a weighted linear combination of the predetermined X's, with $\hat{\Pi}$'s as the weights. Now even if the predetermined variables are truly nonstochastic, the $\hat{\Pi}$'s, being estimators, are stochastic. Therefore, $\hat{Y}_{1t}$ is stochastic too. Now from our discussion of the reduced-form equations and indirect least-squares estimation, it is clear that the reduced-coefficients, the $\hat{\Pi}$'s, are functions of the stochastic disturbances, such as u_2. And since $\hat{Y}_{1t}$ depends on the $\hat{\Pi}$'s, it is likely to be correlated with u_2, which is a component of u_t^*. As a result, $\hat{Y}_{1t}$ is expected to be correlated with u_t^*. But as noted previously, this correlation disappears as the sample size tends to infinity. The upshot of all this is that in small samples the 2SLS procedure may lead to biased estimation.

As this two-stage procedure indicates, the basic idea behind 2SLS is to "purify" the stochastic explanatory variable Y_1 of the influence of the stochastic disturbance u_2. This goal is accomplished by performing the reduced-form regression of Y_1 on all the predetermined variables in the system (Stage 1), obtaining the estimates $\hat{Y}_{1t}$ and replacing Y_{1t} in the original equation by the estimated $\hat{Y}_{1t}$, and then applying OLS to the equation thus transformed (Stage 2). The estimators thus obtained are consistent; that is, they converge to their true values as the sample size increases indefinitely.

To illustrate 2SLS further, let us modify the income–money supply model as follows:

$$Y_{1t} = \beta_{10} + \beta_{12}Y_{2t} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + u_{1t} \tag{20.4.7}$$

$$Y_{2t} = \beta_{20} + \beta_{21}Y_{1t} + \gamma_{23}X_{3t} + \gamma_{24}X_{4t} + u_{2t} \tag{20.4.8}$$

where, in addition to the variables already defined, X_3 = income in the previous time period and X_4 = money supply in the previous period. Both X_3 and X_4 are predetermined.

It can be readily verified that both Eqs. (20.4.7) and (20.4.8) are overidentified. To apply 2SLS, we proceed as follows: In Stage 1 we regress the endogenous variables on *all* the predetermined variables in the system. Thus,

$$Y_{1t} = \hat{\Pi}_{10} + \hat{\Pi}_{11}X_{1t} + \hat{\Pi}_{12}X_{2t} + \hat{\Pi}_{13}X_{3t} + \hat{\Pi}_{14}X_{4t} + \hat{u}_{1t} \tag{20.4.9}$$

$$Y_{2t} = \hat{\Pi}_{20} + \hat{\Pi}_{21}X_{1t} + \hat{\Pi}_{22}X_{2t} + \hat{\Pi}_{23}X_{3t} + \hat{\Pi}_{24}X_{4t} + \hat{u}_{2t} \tag{20.4.10}$$

In Stage 2 we replace Y_1 and Y_2 in the original (structural) equations by their estimated values from the preceding two regressions and then run the OLS regressions as follows:

$$Y_{1t} = \beta_{10} + \beta_{12}\hat{Y}_{2t} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + u^*_{1t} \tag{20.4.11}$$

$$Y_{2t} = \beta_{20} + \beta_{21}\hat{Y}_{1t} + \gamma_{23}X_{3t} + \gamma_{24}X_{4t} + u^*_{2t} \tag{20.4.12}$$

where $u^*_{1t} = u_{1t} + \beta_{12}\hat{u}_{2t}$ and $\hat{u}^*_{2t} = u_{2t} + \beta_{21}\hat{u}_{1t}$. The estimates thus obtained will be consistent.

Note the following features of 2SLS.

1. It can be applied to an individual equation in the system without directly taking into account any other equation(s) in the system. Hence, for solving econometric models involving a large number of equations, 2SLS offers an economical method. For this reason the method has been used extensively in practice.
2. Unlike ILS, which provides multiple estimates of parameters in the overidentified equations, 2SLS provides only one estimate per parameter.
3. It is easy to apply because all one needs to know is the total number of exogenous or predetermined variables in the system without knowing any other variables in the system.
4. Although specially designed to handle overidentified equations, the method can also be applied to exactly identified equations. But then ILS and 2SLS will give identical estimates. (Why?)
5. If the R^2 values in the reduced-form regressions (that is, Stage 1 regressions) are very high, say, in excess of 0.8, the classical OLS estimates and 2SLS estimates will be very close. But this result should not be surprising because if the R^2 value in the first stage is very high, it means that the estimated values of the endogenous variables are very close to their actual values, and hence the latter are less likely to be correlated with the stochastic disturbances in the original structural equations. (Why?)[15] If, however, the

[15]In the extreme case of $R^2 = 1$ in the first-stage regression, the endogenous explanatory variable in the original (overidentified) equation will be practically nonstochastic (why?).

R^2 values in the first-stage regressions are very low, the 2SLS estimates will be practically meaningless because we shall be replacing the original Y's in the second-stage regressions by the estimated $\hat{Y}$'s from the first-stage regressions, which will essentially represent the disturbances in the first-stage regressions. In other words, in this case, the $\hat{Y}$'s will be very poor proxies for the original Y's.

6. Notice that in reporting the ILS regression in Eq. (20.3.15) we did not state the standard errors of the estimated coefficients (for reasons explained in footnote 10). But we can do this for the 2SLS estimates because the structural coefficients are directly estimated from the second-stage (OLS) regressions. There is, however, a caution to be exercised. The estimated standard errors in the second-stage regressions need to be modified because, as can be seen from Eq. (20.4.6), the error term u_t^* is, in fact, the original error term u_{2t} plus $\beta_{21}\hat{u}_t$. Hence, the variance of u_t^* is not exactly equal to the variance of the original u_{2t}. However, the modification required can be easily effected by the formula given in Appendix 20A, Section 20A.2.
7. In using the 2SLS, bear in mind the following remarks of Henri Theil:

> The statistical justification of the 2SLS is of the large-sample type. When there are no lagged endogenous variables, . . . the 2SLS coefficient estimators are consistent if the exogenous variables are constant in repeated samples and if the disturbance[s] [appearing in the various behavioral or structural equations] . . . are independently and identically distributed with zero means and finite variances. . . . If these two conditions are satisfied, the sampling distribution of 2SLS coefficient estimators becomes approximately normal for large samples. . . .
>
> When the equation system contains lagged endogenous variables, the consistency and large-sample normality of the 2SLS coefficient estimators require an additional condition, . . . that as the sample increases the mean square of the values taken by each lagged endogenous variable converges in probability to a positive limit. . . .
>
> If [the disturbances appearing in the various structural equations are] *not* independently distributed, lagged endogenous variables are not independent of the current operation of the equation system . . . , which means these variables are not really predetermined. If these variables are nevertheless treated as predetermined in the 2SLS procedure, the resulting estimators are not consistent.[16]

20.5 2SLS: A Numerical Example

To illustrate the 2SLS method, consider the income–money supply model given previously in Eqs. (20.4.1) and (20.4.2). As shown, the money supply equation is overidentified. To estimate the parameters of this equation, we resort to the two-stage least-squares method. The data required for analysis are given in Table 20.2; this table also gives some data that are required to answer some of the questions given in the exercises.

Stage 1 Regression

We first regress the stochastic explanatory variable income Y_1, represented by GDP, on the predetermined variables private investment X_1 and government expenditure X_2, obtaining the following results:

$$\begin{aligned} \hat{Y}_{1t} &= 2689.848 \quad + \quad 1.8700X_{1t} + \quad 2.0343X_{2t} \\ \text{se} &= \quad (67.9874) \qquad (0.1717) \qquad (0.1075) \\ t &= \quad (39.5639) \qquad (10.8938) \qquad (18.9295) \qquad R^2 = 0.9964 \end{aligned} \tag{20.5.1}$$

[16]Henri Theil, *Introduction to Econometrics,* Prentice Hall, Englewood Cliffs, NJ, 1978, pp. 341–342.

TABLE 20.2
GDP, M2, FEDEXP, TB6, USA, 1970–2005

Source: *Economic Report of the President,* 2007. Tables B-2, B-69, B-84, and B-73.

Observation	GDP (Y_1)	M2 (Y_2)	GPDI (X_1)	FEDEXP (X_2)	TB6 (X_3)
1970	3,771.9	626.5	427.1	201.1	6.562
1971	3,898.6	710.3	475.7	220.0	4.511
1972	4,105.0	802.3	532.1	244.4	4.466
1973	4,341.5	855.5	594.4	261.7	7.178
1974	4,319.6	902.1	550.6	293.3	7.926
1975	4,311.2	1,016.2	453.1	346.2	6.122
1976	4,540.9	1,152.0	544.7	374.3	5.266
1977	4,750.5	1,270.3	627.0	407.5	5.510
1978	5,015.0	1,366.0	702.6	450.0	7.572
1979	5,173.4	1,473.7	725.0	497.5	10.017
1980	5,161.7	1,599.8	645.3	585.7	11.374
1981	5,291.7	1,755.4	704.9	672.7	13.776
1982	5,189.3	1,910.3	606.0	748.5	11.084
1983	5,423.8	2,126.5	662.5	815.4	8.75
1984	5,813.6	2,310.0	857.7	877.1	9.80
1985	6,053.7	2,495.7	849.7	948.2	7.66
1986	6,263.6	2,732.4	843.9	1,006.0	6.03
1987	6,475.1	2,831.4	870.0	1,041.6	6.05
1988	6,742.7	2,994.5	890.5	1,092.7	6.92
1989	6,981.4	3,158.5	926.2	1,167.5	8.04
1990	7,112.5	3,278.6	895.1	1,253.5	7.47
1991	7,100.5	3,379.1	822.2	1,315.0	5.49
1992	7,336.6	3,432.5	889.0	1,444.6	3.57
1993	7,532.7	3,484.0	968.3	1,496.0	3.14
1994	7,835.5	3,497.5	1,099.6	1,533.1	4.66
1995	8,031.7	3,640.4	1,134.0	1,603.5	5.59
1996	8,328.9	3,815.1	1,234.3	1,665.8	5.09
1997	8,703.5	4,031.6	1,387.7	1,708.9	5.18
1998	9,066.9	4,379.0	1,524.1	1,734.9	4.85
1999	9,470.3	4,641.1	1,642.6	1,787.6	4.76
2000	9,817.0	4,920.9	1,735.5	1,864.4	5.92
2001	9,890.7	5,430.3	1,598.4	1,969.5	3.39
2002	10,048.8	5,774.1	1,557.1	2,101.1	1.69
2003	10,301.0	6,062.0	1,613.1	2,252.1	1.06
2004	10,703.5	6,411.7	1,770.6	2,383.0	1.58
2005	11,048.6	6,669.4	1,866.3	2,555.9	3.40

Notes: Y_1 = GDP = gross domestic product (billions of chained 2000 dollars).
Y_2 = M2 = M2 money supply (billions of dollars).
X_1 = GPDI = gross private domestic investment (billions of chained 2000 dollars).
X_2 = FEDEXP = Federal government expenditure (billions of dollars).
X_3 = TB6 = 6-month Treasury bill rate (%).

Stage 2 Regression

We now estimate the money supply function (20.4.2), replacing the endogenous variable Y_1 by Y_1 estimated from Eq. (20.5.1) $(= \hat{Y}_1)$. The results are as follows:

$$\begin{aligned} \hat{Y}_{2t} &= -2440.180 \quad + \quad 0.7920\hat{Y}_{1t} \\ \text{se} &= \quad (127.3720) \qquad (0.0178) \\ t &= \quad (-19.1579) \qquad (44.5246) \qquad R^2 = 0.9831 \end{aligned} \tag{20.5.2}$$

As we pointed out previously, the estimated standard errors given in Eq. (20.5.2) need to be corrected in the manner suggested in Appendix 20.A, Section 20A.2. Effecting this correction (most econometric packages can do it now), we obtain the following results:

$$\begin{aligned} \hat{Y}_{2t} &= -2440.180 \quad + \quad 0.7920\hat{Y}_{1t} \\ \text{se} &= \quad (126.9598) \qquad (0.0212) \\ t &= \quad (-17.3149) \qquad (37.3057) \qquad R^2 = 0.9803 \end{aligned} \tag{20.5.3}$$

As noted in Appendix 20A, Section 20A.2, the standard errors given in Eq. (20.5.3) do not differ much from those given in Eq. (20.5.2) because the R^2 in Stage 1 regression is very high.

OLS Regression

For comparison, we give the regression of money stock on income as shown in Eq. (20.4.2) without "purging" the stochastic Y_{1t} of the influence of the stochastic disturbance term.

$$\begin{aligned} \hat{Y}_{2t} &= -2195.468 \quad + \quad 0.7911Y_{1t} \\ \text{se} &= \quad (126.6460) \qquad (0.0211) \\ t &= \quad (-17.3354) \qquad (37.3812) \qquad R^2 = 0.9803 \end{aligned} \tag{20.5.4}$$

Comparing the "inappropriate" OLS results with the Stage 2 regression, we see that the two regressions are virtually the same. Does this mean that the 2SLS procedure is worthless? Not at all. That in the present situation the two results are practically identical should not be surprising because, as noted previously, the R^2 value in the first stage is very high, thus making the estimated $\hat{Y}_{1t}$ virtually identical with the actual Y_{1t}. Therefore, in this case the OLS and second-stage regressions will be more or less similar. But there is no guarantee that this will happen in every application. An implication, then, is that in overidentified equations one should not accept the classical OLS procedure without checking the second-stage regression(s).

Simultaneity between GDP and Money Supply

Let us find out if GDP (Y_1) and money supply (Y_2) are mutually dependent. For this purpose we use the Hausman test of simultaneity discussed in Chapter 19.

First we regress GDP on X_1 (investment expenditure) and X_2 (government expenditure), the exogenous variables in the system (i.e., we estimate the reduced-form regression). From this regression we obtain the estimated GDP and the residuals $\hat{v}_t$, as suggested in Eq. (19.4.7). Then we regress money supply on estimated GDP and v_t to obtain the following results:

$$\begin{aligned} \hat{Y}_{2t} &= -2198.297 \quad + \quad 0.7915\hat{Y}_{1t} + \; 0.6984\hat{v}_t \\ \text{se} &= \quad (129.0548) \qquad (0.0215) \qquad (0.2970) \\ t &= \quad (-17.0338) \qquad (36.70016) \qquad (2.3511) \end{aligned} \tag{20.5.5}$$

Since the t value of $\hat{v}_t$ is statistically significant (the p value is 0.0263), we cannot reject the hypothesis of simultaneity between money supply and GDP, which should not be surprising. (*Note:* Strictly speaking, this conclusion is valid only in large samples; technically, it is only valid as the sample size increases indefinitely.)

Hypothesis Testing

Suppose we want to test the hypothesis that income has no effect on money demand. Can we test this hypothesis with the usual *t* test from the estimated regression (20.5.2)? Yes, provided the sample is large and provided we correct the standard errors as shown in Eq. (20.5.3), we can use the *t* test to test the significance of an individual coefficient and the *F* test to test joint significance of two or more coefficients, using formula (8.4.7).[17]

What happens if the error term in a structural equation is autocorrelated and/or correlated with the error term in another structural equation in the system? A full answer to this question will take us beyond the scope of the book and is better left for the references (see the reference given in footnote 7). Nevertheless, estimation techniques (such as Zellner's SURE technique) do exist to handle these complications.

To conclude the discussion of our numerical example, it may be added that the various steps involved in the application of 2SLS are now routinely handled by software packages such as STATA and *EViews*. It was only for pedagogical reason we showed the details of 2SLS. See Exercise 20.15.

20.6 Illustrative Examples

In this section we consider some applications of the simultaneous-equation methods.

EXAMPLE 20.1
Advertising, Concentration, and Price Margins

To study the interrelationships among advertising, concentration (as measured by the concentration ratio), and price-cost margins, Allyn D. Strickland and Leonard W. Weiss formulated the following three-equation model.[18]

Advertising intensity function:

$$\text{Ad}/S = a_0 + a_1 M + a_2(\text{CD}/S) + a_3 C + a_4 C^2 + a_5 \text{Gr} + a_6 \text{Dur} \tag{20.6.1}$$

Concentration function:

$$C = b_0 + b_1(\text{Ad}/S) + b_2(\text{MES}/S) \tag{20.6.2}$$

Price-cost margin function:

$$M = c_0 + c_1(K/S) + c_2 \text{Gr} + c_3 C + c_4 \text{GD} + c_5(\text{Ad}/S) + c_6(\text{MES}/S) \tag{20.6.3}$$

where
Ad = advertising expense
S = value of shipments
C = four-firm concentration ratio
CD = consumer demand
MES = minimum efficient scale
M = price/cost margin
Gr = annual rate of growth of industrial production
Dur = dummy variable for durable goods industry
K = capital stock
GD = measure of geographic dispersion of output

[17]But take this precaution: The restricted and unrestricted RSS in the numerator must be calculated using predicted Y (as in Stage 2 of 2SLS) and the RSS in the denominator is calculated using actual rather than predicted values of the regressors. For an accessible discussion of this point, see T. Dudley Wallace and J. Lew Silver, *Econometrics: An Introduction,* Addison–Wesley, Reading, Mass., 1988, Sec. 8.5.

[18]See their "Advertising, Concentration, and Price-Cost Margins," *Journal of Political Economy,* vol. 84, no. 5, 1976, pp. 1109–1121.

EXAMPLE 20.1
(Continued)

By the order conditions for identifiability, Eq. (20.6.2) is overidentified, whereas Eqs. (20.6.1) and (20.6.3) are exactly identified.

The data for the analysis came largely from the 1963 Census of Manufacturers and covered 408 of the 417 four-digit manufacturing industries. The three equations were first estimated by OLS, yielding the results shown in Table 20.3. To correct for the simultaneous-equation bias, the authors reestimated the model using 2SLS. The ensuing results are given in Table 20.4. We leave it to the reader to compare the two results.

TABLE 20.3
OLS Estimates of Three Equations (*t* ratios in parentheses)

	Dependent Variable		
	Ad/*S* **Eq. (20.6.1)**	***C*** **Eq. (20.6.2)**	***M*** **Eq. (20.6.3)**
Constant	−0.0314 (−7.45)	0.2638 (25.93)	0.1682 (17.15)
C	0.0554 (3.56)	—	0.0629 (2.89)
C^2	−0.0568 (−3.38)	—	—
M	0.1123 (9.84)	—	—
CD/*S*	0.0257 (8.94)	—	—
Gr	0.0387 (1.64)		0.2255 (2.61)
Dur	−0.0021 (−1.11)	—	—
Ad/*S*	—	1.1613 (3.3)	1.6536 (11.00)
MES/*S*	—	4.1852 (18.99)	0.0686 (0.54)
K/*S*	—	—	0.1123 (8.03)
GD	—	—	−0.0003 (−2.90)
R^2	0.374	0.485	0.402
df	401	405	401

TABLE 20.4
Two-Stage Least-Squares Estimates of Three Equations (*t* ratios in parentheses)

	Dependent Variable		
	Ad/*S* **Eq. (20.6.1)**	***C*** **Eq. (20.6.2)**	***M*** **Eq. (20.6.3)**
Constant	−0.0245 (−3.86)	0.2591 (21.30)	0.1736 (14.66)
C	0.0737 (2.84)	—	0.0377 (0.93)
C^2	−0.0643 (−2.64)	—	—
M	0.0544 (2.01)	—	—
CD/*S*	0.0269 (8.96)	—	—
Gr	0.0539 (2.09)	—	0.2336 (2.61)
Dur	−0.0018 (−0.93)	—	—
Ad/*S*	—	1.5347 (2.42)	1.6256 (5.52)
MES/*S*	—	4.169 (18.84)	0.1720 (0.92)
K/*S*	—	—	0.1165 (7.30)
GD	—	—	−0.0003 (−2.79)

EXAMPLE 20.2
Klein's Model I

In Example 18.6 we discussed briefly the pioneering model of Klein. Initially, the model was estimated for the period 1920–1941. The underlying data are given in Table 20.5; and OLS, reduced-form, and 2SLS estimates are given in Table 20.6. We leave it to the reader to interpret these results.

(*Continued*)

EXAMPLE 20.2 *(Continued)*

TABLE 20.5 Underlying Data for Klein's Model I

Year	C*	P	W	I	K_{-1}	X	W′	G	T
1920	39.8	12.7	28.8	2.7	180.1	44.9	2.2	2.4	3.4
1921	41.9	12.4	25.5	−0.2	182.8	45.6	2.7	3.9	7.7
1922	45.0	16.9	29.3	1.9	182.6	50.1	2.9	3.2	3.9
1923	49.2	18.4	34.1	5.2	184.5	57.2	2.9	2.8	4.7
1924	50.6	19.4	33.9	3.0	189.7	57.1	3.1	3.5	3.8
1925	52.6	20.1	35.4	5.1	192.7	61.0	3.2	3.3	5.5
1926	55.1	19.6	37.4	5.6	197.8	64.0	3.3	3.3	7.0
1927	56.2	19.8	37.9	4.2	203.4	64.4	3.6	4.0	6.7
1928	57.3	21.1	39.2	3.0	207.6	64.5	3.7	4.2	4.2
1929	57.8	21.7	41.3	5.1	210.6	67.0	4.0	4.1	4.0
1930	55.0	15.6	37.9	1.0	215.7	61.2	4.2	5.2	7.7
1931	50.9	11.4	34.5	−3.4	216.7	53.4	4.8	5.9	7.5
1932	45.6	7.0	29.0	−6.2	213.3	44.3	5.3	4.9	8.3
1933	46.5	11.2	28.5	−5.1	207.1	45.1	5.6	3.7	5.4
1934	48.7	12.3	30.6	−3.0	202.0	49.7	6.0	4.0	6.8
1935	51.3	14.0	33.2	−1.3	199.0	54.4	6.1	4.4	7.2
1936	57.7	17.6	36.8	2.1	197.7	62.7	7.4	2.9	8.3
1937	58.7	17.3	41.0	2.0	199.8	65.0	6.7	4.3	6.7
1938	57.5	15.3	38.2	−1.9	201.8	60.9	7.7	5.3	7.4
1939	61.6	19.0	41.6	1.3	199.9	69.5	7.8	6.6	8.9
1940	65.0	21.1	45.0	3.3	201.2	75.7	8.0	7.4	9.6
1941	69.7	23.5	53.3	4.9	204.5	88.4	8.5	13.8	11.6

*Interpretation of column heads is listed in Example 18.6.

Source: These data are taken from G. S. Maddala, *Econometrics,* McGraw-Hill, New York, 1977, p. 238.

TABLE 20.6* OLS, Reduced-Form and 2SLS Estimates of Klein's Model I

Source: G. S. Maddala, *Econometrics,* McGraw-Hill, New York, 1977, p. 242.

OLS:

$$\hat{C} = \underset{(1.203)}{16.237} + \underset{(0.091)}{0.193}P + \underset{(0.040)}{0.796}(W + W') + \underset{(0.090)}{0.089}P_{-1} \qquad \bar{R}^2 = 0.978 \qquad \text{DW} = 1.367$$

$$\hat{I} = \underset{(5.465)}{10.125} + \underset{(0.097)}{0.479}P + \underset{(0.100)}{0.333}P_{-1} - \underset{(0.026)}{0.112}K_{-1} \qquad \bar{R}^2 = 0.919 \qquad \text{DW} = 1.810$$

$$\hat{W} = \underset{(1.151)}{0.064} + \underset{(0.032)}{0.439}X + \underset{(0.037)}{0.146}X_{-1} + \underset{(0.031)}{0.130}t \qquad \bar{R}^2 = 0.985 \qquad \text{DW} = 1.958$$

Reduced-form:

$$\hat{P} = \underset{(10.870)}{46.383} + \underset{(0.444)}{0.813}P_{-1} - \underset{(0.067)}{0.213}K_{-1} + \underset{(0.252)}{0.015}X_{-1} + \underset{(0.154)}{0.297}t - \underset{(0.385)}{0.926}T + \underset{(0.373)}{0.443}G$$

$$\bar{R}^2 = 0.753 \quad \text{DW} = 1.854$$

$$\widehat{W + W'} = \underset{(8.787)}{40.278} + \underset{(0.359)}{0.823}P_{-1} - \underset{(0.054)}{0.144}K_{-1} + \underset{(0.204)}{0.115}X_{-1} + \underset{(0.124)}{0.881}t - \underset{(0.311)}{0.567}T + \underset{(0.302)}{0.859}G$$

$$\bar{R}^2 = 0.949 \quad \text{DW} = 2.395$$

$$\hat{X} = \underset{(18.860)}{78.281} + \underset{(0.771)}{1.724}P_{-1} - \underset{(0.110)}{0.319}K_{-1} + \underset{(0.438)}{0.094}X_{-1} + \underset{(0.267)}{0.878}t - \underset{(0.669)}{0.565}T + \underset{(0.648)}{1.317}G$$

$$\bar{R}^2 = 0.882 \quad \text{DW} = 2.049$$

2SLS:

$$\hat{C} = \underset{(1.464)}{16.543} + \underset{(0.130)}{0.019}P + \underset{(0.044)}{0.810}(W + W') + \underset{(0.118)}{0.214}P_{-1} \qquad \bar{R}^2 = 0.9726$$

$$\hat{I} = \underset{(8.361)}{20.284} + \underset{(0.191)}{0.149}P + \underset{(0.180)}{0.616}P_{-1} - \underset{(0.040)}{0.157}K_{-1} \qquad \bar{R}^2 = 0.8643$$

$$\hat{W} = \underset{(1.894)}{0.065} + \underset{(0.065)}{0.438}X + \underset{(0.070)}{0.146}X_{-1} + \underset{(0.053)}{0.130}t \qquad \bar{R}^2 = 0.9852$$

*Interpretation of variables is listed in Example 18.6 (standard errors in parentheses).

EXAMPLE 20.3

The Capital Asset Pricing Model Expressed as a Recursive System

In a rather unusual application of recursive simultaneous-equation modeling, Cheng F. Lee and W. P. Lloyd[19] estimated the following model for the oil industry:

$$
\begin{aligned}
R_{1t} &= \alpha_1 + \gamma_1 M_t + u_{1t} \\
R_{2t} &= \alpha_2 + \beta_{21} R_{1t} + \gamma_2 M_t + u_{2t} \\
R_{3t} &= \alpha_3 + \beta_{31} R_{1t} + \beta_{32} R_{2t} + \gamma_3 M_t + u_{3t} \\
R_{4t} &= \alpha_4 + \beta_{41} R_{1t} + \beta_{42} R_{2t} + \beta_{43} R_{3t} + \gamma_4 M_t + u_{4t} \\
R_{5t} &= \alpha_5 + \beta_{51} R_{1t} + \beta_{52} R_{2t} + \beta_{53} R_{3t} + \beta_{54} R_{4t} + \gamma_5 M_t + u_{5t} \\
R_{6t} &= \alpha_6 + \beta_{61} R_{1t} + \beta_{62} R_{2t} + \beta_{63} R_{3t} + \beta_{64} R_{4t} + \beta_{65} R_{5t} + \gamma_6 M_t + u_{6t} \\
R_{7t} &= \alpha_7 + \beta_{71} R_{1t} + \beta_{72} R_{2t} + \beta_{73} R_{3t} + \beta_{74} R_{4t} + \beta_{75} R_{5t} + \beta_{76} R_{6t} + \gamma_7 M_t + u_{7t}
\end{aligned}
$$

where R_1 = rate of return on security 1 (= Imperial Oil)
R_2 = rate of return on security 2 (= Sun Oil)
$\vdots$
R_7 = rate of return on security 7 (= Standard of Indiana)
M_t = rate of return on the market index
u_{it} = disturbances ($i = 1, 2, \ldots, 7$)

Before we present the results, the obvious question is: How do we choose which is security 1, which is security 2, and so on? Lee and Lloyd answer this question purely empirically. They regress the rate of return on security *i* on the rates of return of the remaining six securities and observe the resulting R^2. Thus, there will be seven such regressions. Then they order the estimated R^2 values, from the lowest to the highest. The security having the lowest R^2 is designated as security 1 and the one having the highest R^2 is designated as security 7. The idea behind this is intuitively simple. If the R^2 of the rate of return of, say, Imperial Oil, is lowest with respect to the other six securities, it would suggest that this security is affected least by the movements in the returns of the other securities. Therefore, the causal ordering, if any, runs from this security to the others and there is no feedback from the other securities.

Although one may object to such a purely empirical approach to causal ordering, let us present their empirical results nonetheless, which are given in Table 20.7.

In Exercise 5.5 we introduced the *characteristic line* of modern investment theory, which is simply the regression of the rate of return on security *i* on the market rate of return. The slope coefficient, known as the *beta coefficient,* is a measure of the volatility of the security's return. What the Lee–Lloyd regression results suggest is that there are significant intra-industry relationships between security returns, apart from the common market influence represented by the market portfolio. Thus, Standard of Indiana's return depends not only on the market rate of return but also on the rates of return on Shell Oil, Phillips Petroleum, and Union Oil. To put the matter differently, the movement in the rate of return on Standard of Indiana can be better explained if in addition to the market rate of return we also consider the rates of return experienced by Shell Oil, Phillips Petroleum, and Union Oil.

(*Continued*)

[19]"The Capital Asset Pricing Model Expressed as a Recursive System: An Empirical Investigation," *Journal of Financial and Quantitative Analysis,* June 1976, pp. 237–249.

EXAMPLE 20.3
(Continued)

TABLE 20.7 Recursive System Estimates for the Oil Industry

	Linear Form Dependent Variables						
	Standard of Indiana	Shell Oil	Phillips Petroleum	Union Oil	Standard of Ohio	Sun Oil	Imperial Oil
Standard of Indiana							
Shell Oil	0.2100* (2.859)						
Phillips Petroleum	0.2293* (2.176)	0.0791 (1.065)					
Union Oil	0.1754* (2.472)	0.2171* (3.177)	0.2225* (2.337)				
Standard of Ohio	−0.0794 (−1.294)	0.0147 (0.235)	0.4248* (5.501)	0.1468* (1.735)			
Sun Oil	0.1249 (1.343)	0.1710* (1.843)	0.0472 (0.355)	0.1339 (0.908)	0.0499 (0.271)		
Imperial Oil	−0.1077 (−1.412)	0.0526 (0.6804)	0.0354 (0.319)	0.1580 (1.290)	−0.2541* (−1.691)	0.0828 (0.971)	
Constant	0.0868 (0.681)	−0.0384 (1.296)	−0.0127 (−0.068)	−0.2034 (0.986)	0.3009 (1.204)	0.2013 (1.399)	0.3710* (2.161)
Market index	0.3681* (2.165)	0.4997* (3.039)	0.2884 (1.232)	0.7609* (3.069)	0.9089* (3.094)	0.7161* (4.783)	0.6432* (3.774)
R^2	0.5020	0.4658	0.4106	0.2532	0.0985	0.2404	0.1247
Durbin–Watson	2.1083	2.4714	2.2306	2.3468	2.2181	2.3109	1.9592

*Denotes significance at 0.10 level or better for two-tailed test.
Note: The *t* values appear in parentheses beneath the coefficients.
Source: Cheng F. Lee and W. P. Lloyd, op. cit., Table 3b.

EXAMPLE 20.4
Revised Form of St. Louis Model[20]

The well-known, and often controversial, St. Louis model originally developed in the late 1960s has been revised from time to time. One such revision is given in Table 20.8, and the empirical results based on this revised model are given in Table 20.9. (*Note:* A dot over a variable means the growth rate of that variable.) The model basically consists of Eqs. (1), (2), (4), and (5) in Table 20.8, the other equations representing the definitions. Equation (1) was estimated by OLS. Equations (1), (2), and (4) were estimated using the Almon distributed-lag method with (endpoint) constraints on the coefficients. Where relevant, the equations were corrected for first-order (ρ_1) and/or second-order (ρ_2) serial correlation.

Examining the results, we observe that it is the rate of growth in the money supply that primarily determines the rate of growth of (nominal) GNP and not the rate of growth in high-employment expenditures. The sum of the M coefficients is 1.06, suggesting that a 1 percent (sustained) increase in the money supply on the average leads to about 1.06 percent increase in the nominal GNP. On the other hand, the sum of the E coefficients, about 0.05, suggests that a change in high-employment government expenditure has little impact on the rate of growth of nominal GNP. It is left to the reader to interpret the results of the other regressions reported in Table 20.9.

[20]Federal Reserve Bank of St. Louis, *Review,* May 1982, p. 14.

EXAMPLE 20.4 *(Continued)*

TABLE 20.8 The St. Louis Model

(1) $\dot{Y}_1 = C1 + \sum_{i=0}^{4} CM_i(\dot{M}_{t-i}) + \sum_{i=0}^{4} CE(\dot{E}_{t-i}) + \varepsilon 1_t$

(2) $\dot{P}_t = C2 + \sum_{i=1}^{4} CPE_i(\dot{PE}_{t-i}) + \sum_{i=0}^{5} CD_i(\dot{X}_{t-i} - \dot{XF}^*_{t-i1})$
$+ CPA(\dot{PA}_t) + CDUM1(DUM1) + CDUM2(DUM2) + \varepsilon 2_t$

(3) $\dot{PA}_t = \sum_{i=1}^{21} CPRL_i(\dot{P}_{t-i})$

(4) $RL_t = C3 + \sum_{i=0}^{20} CPRL_i(\dot{P}_{t-i}) + \varepsilon 3_t$

(5) $U_t - UF_t = CG(GAP_t) + CG1(GAP_{t-1}) + \varepsilon 4_t$

(6) $Y_t = (P_t/100)(X_t)$

(7) $\dot{Y}_t = [(Y_t/Y_{t-i})^4 - 1]100$

(8) $\dot{X}_t = [(X_t/X_{t-i})^4 - 1]100$

(9) $\dot{P}_t = [(P_t/P_{t-i})^4 - 1]100$

(10) $GAP_t = [(XF_t/X_t)/XF_t]100$

(11) $\dot{XF}^*_t = [(XF_t/X_{t-1})^4 - 1]100$

Y = nominal GNP
M = money stock (M1)
E = high employment expenditures
P = GNP deflator (1972 = 100)
PE = relative price of energy
X = output in 1972 dollars
XF = potential output (Rasche/Tatom)
RL = corporate bond rate
U = unemployment rate
UF = unemployment rate at full employment
DUM1 = control dummy (1971–III to 1973–I = 1; 0 elsewhere)
DUM2 = postcontrol dummy (1973–II to 1975–I = 1; 0 elsewhere)

Source: Federal Reserve Bank of St. Louis, *Review,* May 1982, p. 14.

TABLE 20.9
In-Sample Estimation: 1960–I to 1980–IV (absolute value of *t* statistic in parentheses)

Source: Federal Reserve Bank of St. Louis, *Review,* May 1982, p. 14.

(1)
$$\begin{aligned}\widehat{\dot{Y}}_t = {} & \underset{(2.15)}{2.44} + \underset{(3.38)}{0.40\dot{M}_t} + \underset{(5.06)}{0.39\dot{M}_{t-1}} + \underset{(2.18)}{0.22\dot{M}_{t-2}} + \underset{(0.82)}{0.06\dot{M}_{t-3}} - \underset{(0.11)}{0.01\dot{M}_{t-4}}\\ & + \underset{(1.46)}{0.06\dot{E}_t} + \underset{(0.63)}{0.02\dot{E}_{t-1}} - \underset{(0.57)}{0.02\dot{E}_{t-2}} - \underset{(0.52)}{0.02\dot{E}_{t-3}} + \underset{(0.34)}{0.01\dot{E}_{t-4}}\end{aligned}$$
$R^2 = 0.39 \quad se = 3.50 \quad DW = 2.02$

(2)
$$\begin{aligned}\widehat{\dot{P}}_t = {} & \underset{(2.53)}{0.96} + \underset{(0.75)}{0.01\dot{PE}_{t-1}} + \underset{(1.96)}{0.04\dot{PE}_{t-2}} - \underset{(0.73)}{0.01\dot{PE}_{t-3}} + \underset{(1.38)}{0.02\dot{PE}_{t-4}}\\ & - \underset{(0.18)}{0.00(\dot{X}_t - \dot{XF}^*_t)} + \underset{(1.43)}{0.01(\dot{X}_{t-1} - \dot{XF}^*_{t-1})} + \underset{(4.63)}{0.02(\dot{X}_{t-2} - \dot{XF}^*_{t-2})}\\ & + \underset{(3.00)}{0.02(\dot{X}_{t-3} - \dot{XF}^*_{t-3})} + \underset{(2.42)}{0.02(\dot{X}_{t-4} - \dot{XF}^*_{t-4}} + \underset{(2.16)}{0.01(\dot{X}_{t-5} - \dot{XF}^*_{t-5})}\\ & + \underset{(10.49)}{1.03(\dot{PA}_t)} - \underset{(1.02)}{0.61(DUM1_t)} + \underset{(2.71)}{1.65(DUM2_t)}\end{aligned}$$
$R^2 = 0.80 \quad se = 1.28 \quad DW = 1.97 \quad \hat{\rho} = 0.12$

(4)
$$\widehat{RL}_t = \underset{(3.12)}{2.97} + \underset{(5.22)}{0.96}\sum_{i=0}^{20} \dot{P}_{t-i}$$
$R^2 = 0.32 \quad se = 0.33 \quad DW = 1.76 \quad \hat{\rho} = 0.94$

(5)
$$\widehat{U_t - UF_t} = \underset{(11.89)}{0.28(GAP_t)} + \underset{(6.31)}{0.14(GAP_{t-1})}$$
$R^2 = 0.63 \quad se = 0.17 \quad DW = 1.95 \quad \hat{\rho}_1 = 1.43 \quad \hat{\rho}_2 = 0.52$

Summary and Conclusions

1. Assuming that an equation in a simultaneous-equation model is identified (either exactly or over-), we have several methods to estimate it.
2. These methods fall into two broad categories: *Single-equation methods* and *systems methods*.
3. For reasons of economy, specification errors, etc., the single-equation methods are by far the most popular. A unique feature of these methods is that one can estimate a single-equation in a multiequation model without worrying too much about other equations in the system. (*Note:* For identification purposes, however, the other equations in the system count.)
4. Three commonly used single-equation methods are **OLS, ILS,** and **2SLS.**
5. Although OLS is, in general, inappropriate in the context of simultaneous-equation models, it can be applied to the so-called **recursive models** where there is a definite but unidirectional cause-and-effect relationship among the endogenous variables.
6. The method of ILS is suited for just or exactly identified equations. In this method OLS is applied to the reduced-form equation, and it is from the reduced-form coefficients that one estimates the original structural coefficients.
7. The method of 2SLS is especially designed for overidentified equations, although it can also be applied to exactly identified equations. But then the results of 2SLS and ILS are identical. The basic idea behind 2SLS is to replace the (stochastic) endogenous explanatory variable by a linear combination of the predetermined variables in the model and use this combination as the explanatory variable in lieu of the original endogenous variable. The 2SLS method thus resembles the **instrumental variable method** of estimation in that the linear combination of the predetermined variables serves as an instrument, or proxy, for the endogenous regressor.
8. A noteworthy feature of both ILS and 2SLS is that the estimates obtained are consistent, that is, as the sample size increases indefinitely, the estimates converge to their true population values. The estimates may not satisfy small-sample properties, such as unbiasedness and minimum variance. Therefore, the results obtained by applying these methods to small samples and the inferences drawn from them should be interpreted with due caution.

EXERCISES

Questions

20.1. State whether each of the following statements is true or false:

a. The method of OLS is not applicable to estimate a structural equation in a simultaneous-equation model.

b. In case an equation is not identified, 2SLS is not applicable.

c. The problem of simultaneity does not arise in a recursive simultaneous-equation model.

d. The problems of simultaneity and exogeneity mean the same thing.

e. The 2SLS and other methods of estimating structural equations have desirable statistical properties only in large samples.

f. There is no such thing as an R^2 for the simultaneous-equation model as a whole.

**g.* The 2SLS and other methods of estimating structural equations are not applicable if the equation errors are autocorrelated and/or are correlated across equations.

h. If an equation is exactly identified, ILS and 2SLS give identical results.

*Optional.

20.2. Why is it unnecessary to apply the two-stage least-squares method to exactly identified equations?

20.3. Consider the following modified Keynesian model of income determination:

$$C_t = \beta_{10} + \beta_{11} Y_t + u_{1t}$$
$$I_t = \beta_{20} + \beta_{21} Y_t + \beta_{22} Y_{t-1} + u_{2t}$$
$$Y_t = C_t + I_t + G_t$$

where C = consumption expenditure
I = investment expenditure
Y = income
G = government expenditure
G_t and Y_{t-1} are assumed predetermined

a. Obtain the reduced-form equations and determine which of the preceding equations are identified (either just or over-).

b. Which method will you use to estimate the parameters of the overidentified equation and of the exactly identified equation? Justify your answer.

20.4. Consider the following results:*

$$OLS\text{: } \widehat{\dot{W}_t} = 0.276 + 0.258\dot{P}_t + 0.046\dot{P}_{t-1} + 4.959V_t \qquad R^2 = 0.924$$
$$OLS\text{: } \widehat{\dot{P}_t} = 2.693 + 0.232\dot{W}_t - 0.544\dot{X}_t + 0.247\dot{M}_t + 0.064\dot{M}_{t-1} \quad R^2 = 0.982$$
$$2SLS\text{: } \widehat{\dot{W}_t} = 0.272 + 0.257\dot{P}_t + 0.046\dot{P}_{t-1} + 4.966V_t \qquad R^2 = 0.920$$
$$2SLS\text{: } \widehat{\dot{P}_t} = 2.686 + 0.233\dot{W}_t - 0.544\dot{X}_t + 0.246\dot{M}_t + 0.046\dot{M}_{t-1} \quad R^2 = 0.981$$

where $\dot{W}_t$, $\dot{P}_t$, $\dot{M}_t$, and $\dot{X}_t$ are percentage changes in earnings, prices, import prices, and labor productivity (all percentage changes are over the previous year), respectively, and where V_t represents unfilled job vacancies (percentage of total number of employees).

"Since the OLS and 2SLS results are practically identical, 2SLS is meaningless." Comment.

†20.5. Assume that production is characterized by the Cobb–Douglas production function

$$Q_i = A K_i^{\alpha} L_i^{\beta}$$

where Q = output
K = capital input
L = labor input
A, α, and β = parameters
i = ith firm

Given the price of final output P, the price of labor W, and the price of capital R, and assuming profit maximization, we obtain the following empirical model of production:

Production function:

$$\ln Q_i = \ln A + \alpha \ln K_i + \beta \ln L_i + \ln u_{1i} \tag{1}$$

*Source: *Prices and Earnings in 1951–1969: An Econometric Assessment,* Department of Employment, United Kingdom, Her Majesty's Stationery Office, London, 1971, p. 30.

†Optional.

Marginal product of labor function:

$$\ln Q_i = -\ln \beta + \ln L_i + \ln \frac{W}{P} + \ln u_{2i} \quad (2)$$

Marginal product of capital function:

$$\ln Q_i = -\ln \alpha + \ln K_i + \ln \frac{R}{P} + \ln u_{3i} \quad (3)$$

where u_1, u_2, and u_3 are stochastic disturbances.

In the preceding model there are three equations in three endogenous variables Q, L, and K. P, R, and W are exogenous.

a. What problems do you encounter in estimating the model if $\alpha + \beta = 1$, that is, when there are constant returns to scale?

b. Even if $\alpha + \beta \neq 1$, can you estimate the equations? Answer by considering the identifiability of the system.

c. If the system is not identified, what can be done to make it identifiable?

Note: Equations (2) and (3) are obtained by differentiating Q with respect to labor and capital, respectively, setting them equal to W/P and R/P, transforming the resulting expressions into logarithms, and adding (the logarithm of) the disturbance terms.

20.6. Consider the following demand-and-supply model for money:

Demand for money: $M_t^d = \beta_0 + \beta_1 Y_1 + \beta_2 R_t + \beta_3 P_t + u_{1t}$

Supply of money: $M_t^s = \alpha_0 + \alpha_1 Y_t + u_{2t}$

where M = money
Y = income
R = rate of interest
P = price

Assume that R and P are predetermined.

a. Is the demand function identified?

b. Is the supply function identified?

c. Which method would you use to estimate the parameters of the identified equation(s)? Why?

d. Suppose we modify the supply function by adding the explanatory variables Y_{t-1} and M_{t-1}. What happens to the identification problem? Would you still use the method you used in (*c*)? Why or why not?

20.7. Refer to Exercise 18.10. For the two-equation system there obtain the reduced-form equations and estimate their parameters. Estimate the indirect least-squares regression of consumption on income and compare your results with the OLS regression.

Empirical Exercises

20.8. Consider the following model:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + u_{1t}$$
$$Y_t = \alpha_0 + \alpha_1 R_t + u_{2t}$$

where M_t (money supply) is exogenous, R_t is the interest rate, and Y_t is GDP.

a. How would you justify the model?

b. Are the equations identified?

c. Using the data given in Table 20.2, estimate the parameters of the identified equations. Justify the method(s) you use.

20.9. Suppose we change the model in Exercise 20.8 as follows:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + \beta_3 Y_{t-1} + u_{1t}$$
$$Y_t = \alpha_0 + \alpha_1 R_t + u_{2t}$$

a. Find out if the system is identified.

b. Using the data given in Table 20.2, estimate the parameters of the identified equation(s).

20.10. Consider the following model:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + u_{1t}$$
$$Y_t = \alpha_0 + \alpha_1 R_t + \alpha_2 I_t + u_{2t}$$

where the variables are as defined in Exercise 20.8. Treating I (domestic investment) and M exogenously, determine the identification of the system. Using the data given in Table 20.2, estimate the parameters of the identified equation(s).

20.11. Suppose we change the model of Exercise 20.10 as follows:

$$R_t = \beta_0 + \beta_1 M_t + \beta_2 Y_t + u_{1t}$$
$$Y_t = \alpha_0 + \alpha_1 R_t + \alpha_2 I_t + u_{2t}$$
$$I_t = \gamma_0 + \gamma_1 R_t + u_{3t}$$

Assume that M is determined exogenously.

a. Find out which of the equations are identified.

b. Estimate the parameters of the identified equation(s) using the data given in Table 20.2. Justify your method(s).

20.12. Verify the standard errors reported in Eq. (20.5.3).

20.13. Return to the demand-and-supply model given in Eqs. (20.3.1) and (20.3.2). Suppose the supply function is altered as follows:

$$Q_t = \beta_0 + \beta_1 P_{t-1} + u_{2t}$$

where P_{t-1} is the price prevailing in the previous period.

a. If X (expenditure) and P_{t-1} are predetermined, is there a simultaneity problem?

b. If there is, are the demand and supply functions each identified? If they are, obtain their reduced-form equations and estimate them from the data given in Table 20.1.

c. From the reduced-form coefficients, can you derive the structural coefficients? Show the necessary computations.

20.14. *Class Exercise:* Consider the following simple macroeconomic model for the U.S. economy, say, for the period 1960–1999.*

Private consumption function:

$$C_t = \alpha_0 + \alpha_1 Y_t + \alpha_2 C_{t-1} + u_{1t} \qquad \alpha_1 > 0, 0 < \alpha_2 < 1$$

Private gross investment function:

$$I_t = \beta_0 + \beta_1 Y_t + \beta_2 R_t + \beta_3 I_{t-1} + u_{2t} \qquad \beta_1 > 0, \beta_2 < 0, 0 < \beta_3 < 1$$

A money demand function:

$$R_t = \lambda_0 + \lambda_1 Y_t + \lambda_2 M_{t-1} + \lambda_3 P_t + \lambda_4 R_{t-1} + u_{3t}$$
$$\lambda_1 > 0, \lambda_2 < 0, \lambda_3 > 0, 0 < \lambda_4 < 1$$

*Adapted from H. R. Seddighi, K. A. Lawler, and A. V. Katos, *Econometrics: A Practical Approach*, Routledge, New York, 2000, p. 204.

Income identity:

$$Y_t = C_t + I_t + G_t$$

where C = real private consumption; I = real gross private investment, G = real government expenditure, Y = real GDP, M = M2 money supply at current prices, R = long-term interest rate (%), and P = Consumer Price Index. The endogenous variables are C, I, R, and Y. The predetermined variables are: $C_{t-1}, I_{t-1}, M_{t-1}, P_t, R_{t-1}$, and G_t plus the intercept term. The u's are the error terms.

a. Using the order condition of identification, determine which of the four equations are identified, either exact or over-.

b. Which method(s) do you use to estimate the identified equations?

c. Obtain suitable data from government and/or private sources, estimate the model, and comment on your results.

20.15. In this exercise we examine data for 534 workers obtained from the Current Population Survey (CPS) for 1985. The data can be found as Table 20.10 on the textbook website.* The variables in this table are defined as follows:

W = wages \$, per hour; occup = occupation; sector = 1 for manufacturing, 2 for construction, 0 for other; union = 1 if union member, 0 otherwise; educ = years of schooling; exper = work experience in years; age = age in years; sex = 1 for female; marital status = 1 if married; race = 1 for other, 2 for Hispanic, 3 for white; region = 1 if lives in the South.

Consider the following simple wage determination model:

$$\ln W = \beta_1 + \beta_2 \text{Educ} + \beta_3 \text{Exper} + \beta_4 \text{Exper}^2 + u_i \qquad (1)$$

a. Suppose education, like wages, is endogenous. How would you find out that in Equation (1) education is in fact endogenous? Use the data given in the table in your analysis.

b. Does the Hausman test support your analysis in (*a*)? Explain fully.

20.16. *Class Exercise:* Consider the following demand-and-supply model for loans of commercial banks to businesses:

$$\textit{Demand: } Q_t^d = \alpha_1 + \alpha_2 R_t + \alpha_2 \text{RD}_t + \alpha_4 \text{IPI}_t + u_{1t}$$

$$\textit{Supply: } Q_t^s = \beta_1 + \beta_2 R_t + \beta_3 \text{RS}_t + \beta_4 \text{TBD}_t + u_{2t}$$

Where Q = total commercial bank loans (\$billion); R = average prime rate; RS = 3-month Treasury bill rate; RD = AAA corporate bond rate; IPI = Index of Industrial Production; and TBD = total bank deposits.

a. Collect data on these variables for the period 1980–2007 from various sources, such as www.economagic.com, the website of the Federal Reserve Bank of St. Louis, or any other source.

b. Are the demand and supply functions identified? List which variables are endogenous and which are exogenous.

c. How would you go about estimating the demand and supply functions listed above? Show the necessary calculations.

d. Why are both R and RS included in the model? What is the role of IPI in the model?

*Data can be found on the Web, at http://lib.stat.cmu.edu/datasets/cps_85_wages.

Appendix 20A

20A.1 Bias in the Indirect Least-Squares Estimators

To show that the ILS estimators, although consistent, are biased, we use the demand-and-supply model given in Eqs. (20.3.1) and (20.3.2). From Eq. (20.3.10) we obtain

$$\hat{\beta}_1 = \frac{\hat{\Pi}_3}{\hat{\Pi}_1}$$

Now

$$\hat{\Pi}_3 = \frac{\sum q_t x_t}{\sum x_t^2} \qquad \text{from Eq. (20.3.7)}$$

and

$$\hat{\Pi}_1 = \frac{\sum p_t x_t}{\sum x_t^2} \qquad \text{from Eq. (20.3.5)}$$

Therefore, on substitution, we obtain

$$\hat{\beta}_1 = \frac{\sum q_t x_t}{\sum p_t x_t} \tag{1}$$

Using Eqs. (20.3.3) and (20.3.4), we obtain

$$p_t = \Pi_1 x_t + (w_t - \bar{w}) \tag{2}$$

$$q_t = \Pi_3 x_t + (v_t - \bar{v}) \tag{3}$$

where $\bar{w}$ and $\bar{v}$ are the mean values of w_t and v_t, respectively.

Substituting Eqs. (2) and (3) into Eq. (1), we obtain

$$\begin{aligned} \hat{\beta}_1 &= \frac{\Pi_3 \sum x_t^2 + \sum (v_t - \bar{v}) x_t}{\Pi_1 \sum x_t^2 + \sum (w_t - \bar{w}) x_t} \\ &= \frac{\Pi_3 + \sum (v_t - \bar{v}) x_t / \sum x_t^2}{\Pi_1 + \sum (w_t - \bar{w}) x_t / \sum x_t^2} \end{aligned} \tag{4}$$

Since the expectation operator E is a linear operator, we cannot take the expectation of Eq. (4), although it is clear that $\hat{\beta}_1 \neq (\Pi_3/\Pi_1)$ generally. (Why?)

But as the sample size tends to infinity, we can obtain

$$\text{plim}\,(\hat{\beta}_1) = \frac{\text{plim}\,\Pi_3 + \text{plim}\,\sum (v_t - \bar{v}) x_t / \sum x_t^2}{\text{plim}\,\Pi_1 + \text{plim}\,\sum (w_t - \bar{w}) x_t / \sum x_t^2} \tag{5}$$

where use is made of the properties of plim, namely, that

$$\text{plim}\,(A + B) = \text{plim}\,A + \text{plim}\,B \text{ and plim}\left(\frac{A}{B}\right) = \frac{\text{plim}\,A}{\text{plim}\,B}$$

Now as the sample size is increased indefinitely, the second term in both the denominator and the numerator of Eq. (5) tends to zero (why?), yielding

$$\text{plim}\,(\hat{\beta}_1) = \frac{\Pi_3}{\Pi_1} \tag{6}$$

showing that, although biased, $\hat{\beta}_1$ is a consistent estimator of β_1.

20A.2 Estimation of Standard Errors of 2SLS Estimators

The purpose of this appendix is to show that the standard errors of the estimates obtained from the second-page regression of the 2SLS procedure, using the formula applicable in OLS estimation, are not the "proper" estimates of the "true" standard errors. To see this, we use the income–money supply model given in Eqs. (20.4.1) and (20.4.2). We estimate the parameters of the overidentified money supply function from the second-stage regression as

$$Y_{2t} = \beta_{20} + \beta_{21}\hat{Y}_{1t} + u_t^* \qquad \textbf{(20.4.6)}$$

where

$$u_t^* = u_{2t} + \beta_{21}\hat{u}_t \qquad \textbf{(7)}$$

Now when we run regression (20.4.6), the standard error of, say, $\hat{\beta}_{21}$ is obtained from the following expression:

$$\operatorname{var}(\hat{\beta}_{21}) = \frac{\hat{\sigma}_{u^*}^2}{\sum \hat{y}_{1t}^2} \qquad \textbf{(8)}$$

where

$$\hat{\sigma}_{u^*}^2 = \frac{\sum(\hat{u}_t^*)^2}{n-2} = \frac{\sum(Y_{2t} - \hat{\beta}_{20} - \hat{\beta}_{21}\hat{Y}_{1t})^2}{n-2} \qquad \textbf{(9)}$$

But $\sigma_{u^*}^2$ is not the same thing as $\hat{\sigma}_{u_2}^2$, where the latter is an unbiased estimate of the true variance of u_2. This difference can be readily verified from Eq. (7). To obtain the true (as defined previously) $\hat{\sigma}_{u_2}^2$, we proceed as follows:

$$\hat{u}_{2t} = Y_{2t} - \hat{\beta}_{20} - \hat{\beta}_{21}Y_{1t}$$

where $\hat{\beta}_{20}$ and $\hat{\beta}_{21}$ are the estimates from the second-stage regression. Hence,

$$\hat{\sigma}_{u_2}^2 = \frac{\sum(Y_{2t} - \hat{\beta}_{20} - \hat{\beta}_{21}Y_{1t})^2}{n-2} \qquad \textbf{(10)}$$

Note the difference between Eqs. (9) and (10): In Eq. (10) we use actual Y_1 rather than the estimated Y_1 from the first-stage regression.

Having estimated Eq. (10), the easiest way to correct the standard errors of coefficients estimated in the second-stage regression is to multiply each one of them by $\hat{\sigma}_{u_2}/\hat{\sigma}_{u^\ddagger}$. Note that if Y_{1t} and $\hat{Y}_{1t}$ are very close, that is, the R^2 in the first-stage regression is very high, the correction factor $\hat{\sigma}_{u_2}/\hat{\sigma}_{u^*}$ will be close to 1, in which case the estimated standard errors in the second-stage regression may be taken as the true estimates. But in other situations, we shall have to use the preceding correction factor.